MODERN SIGNALS AND SYSTEMS

HUIBERT KWAKERNAAK
RAPHAEL SIVAN

This book contains an elementary, well-integrated, and comprehensive exploration of the basics of signal theory, and of both the time- and frequency-domain analyses of systems. You will find a wealth of examples illustrating practical applications.

The discrete and continuous-time cases are presented in parallel, at times in a two-column format for easy comparison and understanding. The book also offers coverage of:

- Linear systems
- Stability of convolution systems
- Harmonic and periodic inputs
- Frequency response of various system models
- State description of systems
- Expansion theory
- Fourier series analysis and transforms
- The z- and Laplace transforms

To put the material into a real-world context, the book examines applications in separate chapters of signal processing, digital filtering, communication systems, and automatic control systems.

Included with each book is application software containing a powerful interpreter named SIGSYS that offers a wide and flexible range of operations to generate and handle signals, such as Fourier transformation, convolution, and the integration of differential equations. In addition, graphical support provides instantaneous visualization.

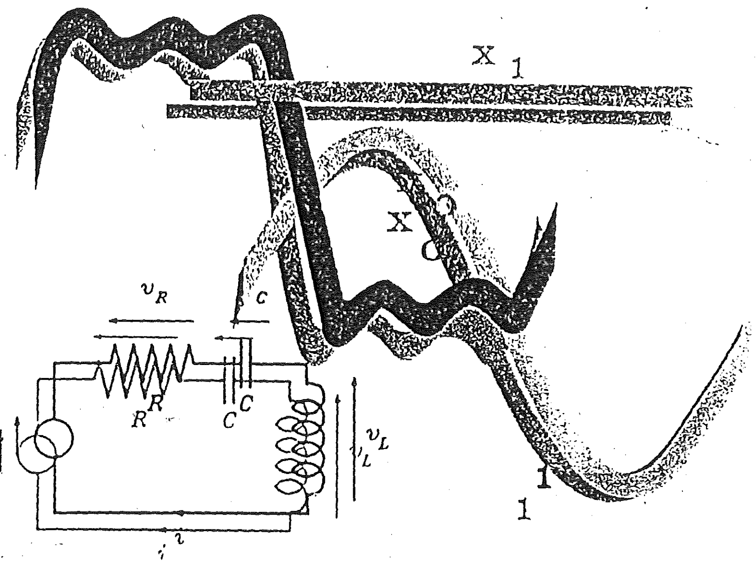
ISBN 0-13-809252-4
90000
9 780138 092528

PRENTICE HALL
Englewood Cliffs, N.J. 07632

KWAKERNAAK
SIVAN
MODERN SIGNALS AND SYSTEMS
DISK INCLUDED
PRENTICE HALL

DISK INCLUDED

MODERN SIGNALS AND SYSTEMS



HUIBERT KWAKERNAAK
RAPHAEL SIVAN

PRENTICE HALL INFORMATION AND SYSTEM SCIENCES SERIES
Thomas Kailath, Series Editor

PRENTICE HALL INFORMATION
AND SYSTEM SCIENCES SERIES
Thomas Kailath, Editor

ɪDERSON & MOORE	*Optimal Control: Linear Quadratic Methods*
ɪDERSON & MOORE	*Optimal Filtering*
ɾROM & WITTENMARK	*Computer-Controlled Systems: Theory and Design, 2/E*
YD & BARRATT	*Linear Controller Design*
RDNER	*Statistical Spectral Analysis: A Nonprobabilistic Theory*
ɔDWIN & SIN	*Adaptive Filtering, Prediction, and Control*
ʌY & DAVISSON	*Random Processes: A Mathematical Approach for Engineers*
KIN	*Adaptive Filter Theory*
	Fundamentals of Digital Image Processing
ɪSON	*Lectures on Adaptive Parameter Estimation*
ATH	*Linear Systems*
ʌR & VARAIYA	*Stochastic Systems: Estimation, Identification, and Adaptive Control*
ɪ	*VLSI Array Processors*
·, WHITEHOUSE, & ɪLATH, EDS.	*VLSI and Modern Signal Processing*
ERNAAK & SIVAN	*Modern Signals and Systems*
ʌU	*System Identification and Control Design Using PIM+ Software*
	System Identification: Theory for the User
ɪSKI	*Medical Imaging Systems*
ETON & GOODWIN	*Digital Control and Estimation: A Unified Approach*
DRA & ʌSWAMY	*Stable Adaptive Systems*
& BODSON	*Adaptive Control: Stability, Convergence, and Robustness*
ɴ & SRINATH	*Continuous and Discrete Signals and Systems*
	Digital Communications by Satellite
ɪS	*Designing Digital Filters*

MODERN SIGNALS AND SYSTEMS

Huibert Kwakernaak
University of Twente

Raphael Sivan
Technion, Israel Institute of Technology

with software by
Rens C. W. Strijbos
University of Twente

PRENTICE HALL, Englewood Cliffs, NJ 07632

MODERN SIGNALS AND SYSTEMS

Library of Congress Cataloging-in-Publication Data

KWAKERNAAK, HUIBERT.
Modern signals and systems/Huibert Kwakernaak, Raphael Sivan; with software by Rens C. W. Strijbos.
p. cm.
Includes bibliographical references (p.)
ISBN 0-13-809252-4
1. Signal processing. 2. System analysis. I. Sivan, Raphael. II. Strijbos, Rens C. W. III. Title.
TK5102.5.K895 1991
621.382'2—dc20 89-39977
CIP

Editorial/production supervision and interior design: Joseph Scordato
Cover design: Annemarie Kwakernaak
Series logo design: A.M. Bruckstein
Manufacturing buyers: Lori Bulwin/Patrice Fraccio

Printed in the United States of America

10 9 8 7 6 5 4 3 2

ISBN 0-13-809252-4

PRENTICE-HALL INTERNATIONAL (UK) LIMITED, *London*
PRENTICE-HALL OF AUSTRALIA PTY. LIMITED, *Sydney*
PRENTICE-HALL CANADA INC., *Toronto*
PRENTICE-HALL HISPANOAMERICANA, S.A., *Mexico*
PRENTICE-HALL OF INDIA PRIVATE LIMITED, *New Delhi*
PRENTICE-HALL OF JAPAN, INC., *Tokyo*
SIMON & SCHUSTER ASIA PTE. LTD., *Singapore*
EDITORA PRENTICE-HALL DO BRASIL, LTDA., *Rio de Janeiro*

To my parents
H. K.

To Ilana, Ori, Ayelet, Keren, and Yael
R. S.

Contents

Preface

Almost twenty years ago, as young and inexperienced scientists, we dared (wh did we take the chutzpah?) to write an advanced graduate textbook, *Linear Optin Control Systems* (New York: John Wiley, 1972). We found the collaboration enj able and rewarding. A few years ago, being not so young anymore, and possil more mature, we decided to join hands again, this time in what we thought to be more modest endeavor, to write an undergraduate textbook on modern signals a systems.

Having taught this material over and over again, we had little doubt that would not be a too difficult or lengthy task to organize what we thought to be tl Signals and Systems material. How wrong we were! We ignored the fact that tin and again, while standing in front of our classes, we experienced a moment of hes tation, realizing that not all the material we were teaching was totally consistent an completely clear.

It took us nearly four years of work, and almost endless brain racking and sou searching, to integrate the signals and systems material into a unified framework an to produce a book we are satisfied with. We hope that the meticulous care we tool in creating the text shows. We also hope that teachers and students alike appreciat the concise and precise style to which we aspired without wanting to compromise the intuitive appeal.

Our *Modern Signals and Systems* is a textbook for a one- or two-semester junior course, which often goes by the name "Signals and Systems", or sometimes by names such as "Linear Systems" or "Dynamical Systems," and is a required course in the curriculum of most electrical engineering departments. The book contains a comprehensive and well-integrated treatment of the basic notions of signal theory and of both the time- and frequency-domain analysis of systems. The discrete- and continuous-time case are treated in parallel, sometimes even typographically in two column format. The book contains many examples, several of which are pursued over many chapters. An extensive collection of homework problems concludes each chapter. In addition, the final section of each chapter provides problems whose solution requires a computer. An unusual feature is that the book comes complete with application software of near-professional quality, provided on a disk, that runs on a personal computer.

It is assumed that the student has a background in basic calculus and algebra, knows how to work with complex numbers, has heard of differential equations and linear algebra, knows the fundamentals of physics and electricity, and possibly has had an introductory course in electrical circuits. To use the software, some familiarity with a personal computer is helpful.

From this book we have taught two-quarter (at the University of Twente) and one-semester (at the Technion) courses to sophomore students. We managed to cover practically all of Chapters 1–8 and a sprinkle of the applications in Chapters 9–11. In departments that devote two semesters to the Signals and Systems course, the entire book may easily be covered. There may even be time to pursue the applications dealt with in Chapters 9–11 in a little more depth, at the discretion of the teacher.

A chapter-by-chapter description of the material follows.

Chapter 1 offers a brief overview of the ideas of signals and systems. By way of motivation, a sketch is given of the application areas that are elaborated in the final three chapters of the book.

Chapter 2 presents an introduction to signals. We describe basic notions such as the time axis of signals, discrete- and continuous-time signals, periodic and harmonic signals, various operations on signals, and signal spaces. The final section is devoted to *generalized* signals. A more elaborate treatment of this material, based on *distribution* theory, is presented in Supplement C.

Chapter 3 deals with a number of fundamental ideas related to systems. Two types of systems are introduced at this point: *input-output* systems and *input-output mapping* systems. The definitions of these systems are *set theoretic*, based on the approach of Jan C. Willems to system theory. These ideas allow an axiomatic approach to system theory even at the level of an undergraduate text. The chapter continues by distinguishing various types of systems. By way of linearity and time-invariance, the discussion arrives at *convolution* systems. The convolution operation is thoroughly treated, and the stability of convolution systems is touched upon. A study of the response of convolution systems to harmonic inputs results in the *frequency response* function. The chapter concludes with a discussion of the response of convolution systems to periodic inputs—leading to *cyclical* convolution—and a brief introduction to the interconnection of systems.

In Chapter 4 we consider systems described by constant coefficient linear difference and differential equations. After a review of elementary material concerning the solution of constant coefficient linear difference and differential equations, we deal with the impulse response, stability, and frequency response of these systems. A central notion is that of the *initially-at-rest system*. In the discussion of stability we introduce *bounded-input-bounded-output* (BIBO) and *converging-input-converging-output* (CICO) stability.

Chapter 5 deals with the state description of systems. Also, here the initial approach is set theoretic. The definition of the state rests upon the *state matching property* formulated by Willems. After introducing the basic ideas, the realization of linear difference and differential systems as state systems is discussed. A further section is devoted to the existence of solutions of state equations and the numerical integration of state differential equations. The chapter continues with the explicit solution of linear state equations and modal analysis. The treatment of the stability of state systems is integrated with that of difference and differential systems.

Chapters 6–8 are devoted to the *frequency domain* description of signals and systems. Chapter 6 begins with a concise but fundamental presentation of signal *expansion*. The theory is made concrete by the introduction of *harmonic* bases for signal spaces and the associated finite and infinite *Fourier series expansions*. Various aspects of these expansions, such as the identities of Plancherel and Parseval, convergence properties, the trigonometric form, symmetry properties, and generalized infinite Fourier series, are carefully presented. The chapter concludes with a treatment of the response of convolution systems to periodic inputs.

Chapter 7 is devoted to Fourier *transforms*. First, transform theory is explained in an abstract setting. The discussion soon focuses on *expansion transforms* (i.e., the transformation from a signal to its expansion coeffcients). In this way, the Fourier series expansions of finite-time and periodic signals of Chapter 6 immediately lead to two of the four Fourier transforms that are considered, namely, the discrete-to-discrete Fourier transform (abbreviated DDFT, more conventionally known as the discrete Fourier transform) and the continuous-to-discrete Fourier transform (CDFT). The properties of these transforms are reviewed. Following this it is shown that the expansion of aperiodic rather than periodic or finite-time signals leads to *Fourier integral expansions*. These result in the discrete-to-continuous Fourier transform (DCFT, also known as the discrete-time Fourier transform) and the continuous-to-continuous Fourier transform (CCFT, commonly known as the Fourier integral transform). Also, the properties of these last two transforms are discussed in detail, emphasizing parallels. The chapter ends with showing how Fourier transform theory is used in the frequency domain analysis of convolution systems.

The third chapter dealing with frequency domain analysis, Chapter 8, is devoted to the z-transform and the Laplace transform. First, it is explained that Fourier transform theory cannot handle exponentially increasing signals. To overcome this difficulty, the DCFT and CCFT are modified to the (two-sided) z-transform and Laplace transform. To deal with initial value problems, the one-sided z-transform and Laplace transform are introduced. The existence, properties and inversion of the z-transform and Laplace transform are treated with considerable completeness. Three separate sections of Chapter 8 are devoted to the application of these transforms to the analysis of convolution systems, difference and differential systems, and state systems.

The final chapters of the book present applications of the theory to three important areas for which a course on Signals and Systems forms a prerequisite: signal processing and digital filtering, communication, and feedback and automatic control. Indeed, teachers of courses on these subjects may find that part of their material is covered in this book.

Chapter 9 is the first of the applications chapters. It concerns signal processing and digital filtering. The chapter begins with a discussion of the effect of sampling and interpolation on the frequency content of signals. This leads to a transparent derivation of the sampling theorem. Next on- and off-line signal processing are introduced. After a discussion of windows and windowing, two sections are devoted to an elementary treatment of various methods to design finite and infinite impulse response digital filters. The chapter ends with a derivation of the fast Fourier transform and some considerations about the numerical computation of transforms and convolutions.

Chapter 10 outlines some basic applications to communication theory. The chapter begins with the application of Fourier transforms to the description of narrow-band signals. This is a beautiful theory, which is indispensable in the analysis of modulation and demodulation. Following this, various well-known modulation schemes, including amplitude and frequency modulation, are presented. The chapter ends with a brief discussion of multiplexing.

Chapter 11, finally, contains a concise outline of feedback theory and automatic control. The potential benefits of feedback are demonstrated at a fairly abstract level, illustrated by simple concrete examples. The final section of this chapter reviews various important results on the stability of feedback systems.

The body of the text is complemented with five supplements.

As was mentioned earlier, the book comes with computer software, provided on a disk. The software consists of a powerful interpreter named SIGSYS, which offers a wide and flexible range of operations to generate and handle signals, including Fourier tranformation, convolution, and the integration of differential equations. The software has been strongly inspired by MATLAB, which is a widely used computational tool. The difference is that where the main data type of MATLAB is a matrix, that of SIGSYS is a real- or complex-valued *signal* In addition, complex and real scalars and polynomials are supported. SIGSYS offers an opportunity to do calculations of a widely varying nature interactively. The graphic support provides instantaneous visualization. An extensive Tutorial, describing and illustrating all the operations and commands of SIGSYS, is included at the end of the book. The READ.ME file on the disk that contains SIGSYS should be consulted before running the program. It contains instructions how to set up and start SIGSYS and also lists corrections and modifications to the Tutorial. The disk furthermore contains a number of demos.

We have used SIGSYS for several years to run a laboratory course in parallel with the Signals and Systems course as follows. During the first weeks of the term, students were instructed to go through the first 16 sections of the Tutorial and to do all the exercises provided by the Tutorial. In addition, the students were assigned two or three Computer Exercises from Chapters 2 and 3, whose main purpose is to become familiar with the computer. Depending on the available time, during the rest of the course students were assigned a number of other problems from the Computer Exercises for the rest of the chapters.

A solutions manual for the Problems and Computer Exercises is ava teachers who adopt the book and may be obtained from the publisher. The comes with a disk that provides solutions to all the Computer Exercises i form. Also, hard-copy solutions are supplied.

The story of the software is this: Right at the initial stages of plann book it occurred to us that supporting the text with software would be timely structive. The success of MATLAB was a powerful stimulus, and it stood e for SIGSYS in many ways. We were lucky to get Rens Strijbos, a long-time ate, to develop the program for us. The time and effort spent on this work a comparable with our own exertions. The software was written in the C langua developed in a UNIX environment by using a variety of programming tools.

Writing this book also got us in other ways involved in high technolog early versions of the text were prepared by using the text formatting language and the figures were produced with a drawing program. Being able to keep i touch by electronic mail was a forceful incentive. Fax and international couric from time to time supplemented communication by ordinary mail. The work book took us on numerous trips back and forth between Israel and The Nethe and also to such places as Amherst, Mass., Berlin, Bern, Englewood Cliffs, and, last but not least, the Sinai desert.

We are grateful to our teachers, in particular Lotfi Zadeh and Charlie De who showed us what perspicuity means, and to our students, who keep insisti getting things clear. Our departments deserve credit for accepting our absorpti the project of writing the book, our periods of absence, footing the bills for puter time, and supporting part of the traveling. In particular, we acknowledg support of the Fund for the Promotion of Research of the Technion. We thank secretaries, Marja Langkamp and Annette Berg, for their invaluable help. In co sion, we wish to express our sincere appreciation to Tim Bozik of Prentice Hal his continued belief in the project even when we started missing deadlines.

Enschede and Ha

Huibert Kwakern
Raphael Sivan

1

Overview of Signals and Systems

1.1 INTRODUCTION

This textbook is an introductory study of signals and systems. As we develop the subject, we encounter various kinds of signals: discrete-time and continuous-time signals, real- and complex-valued signals, signals of finite and infinite duration, and many more. We also meet a variety of systems: input-output systems, input-output mapping systems, input-output-state systems, linear systems, time-invariant systems, each in discrete- and continuous-time versions. All these notions will be precisely defined as the need arises. In this chapter we first present somewhat loose explanations of what is meant by the terms signal and system, illustrate the definitions with examples, and describe some areas where the theory of signals and systems is applied.

Signals

A *signal,* roughly, is a phenomenon, arising in some environment, that may be described quantitatively. Examples of signals to be discussed in this book are electrical signals, such as electrical voltages or currents in an electrical circuit. Other signals are auditive signals, visual signals, and sequences of bits that emanate from computers.

Systems

A *system* is, more or less, any part of an environment that causes certain signals that exist in that environment to be related. An electrical circuit is a typical example of a system, because the voltages and currents that exist within the circuit are related.

The signals associated with an electrical circuit are not only voltages and currents but are also magnetic fluxes and electrical charges. The signals that go with a mechanical system, such as a car that moves along a highway, are positions and velocities. The signals associated with the national economy of a country, to mention a system of a completely different nature, are quantities such as national income and expenditure, labor force, and capital equipment.

The signals associated with a system are not arbitrary but are interrelated as a result of the internal mechanisms of the system. The electrical voltages and currents in an electrical circuit, for instance, are interrelated because of the electromagnetic laws. The interrelation of the signals imposed by the laws that govern the system is called the *rule* of the system.

Rather than pursuing in this introductory chapter the abstractions of the notions of signals and systems, we devote the chapter to illustrations. In the next section we describe an assortment of systems and associated signals. Next, in Section 1.3, we present some engineering problems whose solution requires a theory of signals and systems. These problems are drawn from the areas of *signal processing*, *communication engineering*, and *automatic control*.

1.2 EXAMPLES OF SIGNALS AND SYSTEMS

In this section, various examples of signals and systems are presented.

1.2.1. Example: Resistive electrical circuit element. Consider a resistive electrical circuit element with two ports as in Fig. 1.1. The signals associated with

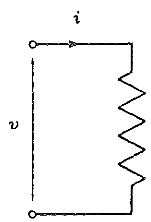


Figure 1.1 A resistive electrical circuit element.

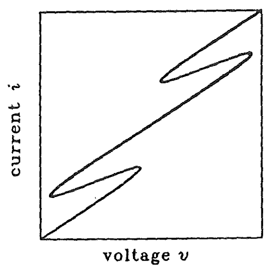


Figure 1.2 Voltage-current characteristic.

the system are the *voltage* v across the circuit element and the *current* i through it. Both the voltage v and the current i are characterized by a real number. The relation between v and i, called the *voltage-current characteristic*, is given by a graph $f(v, i) = 0$, like the one in Fig. 1.2. This graph represents the *rule* of the system: The signals associated with the system are constrained to those voltage-current pairs (v, i) that lie on the graph. Note that for the graph of Fig. 1.2, neither the voltage v determines the current i uniquely nor does the current i uniquely prescribe the voltage v. ∎

Input-Output Systems

Often, though not always, one may designate some of the signals associated with the system as *input* signals, through which the environment influences the system, and some as *output* signals, by which, in turn, the system affects the environment. Figure 1.3 illustrates the situation. Systems of this type are called *input-output* systems. Note that we do *not* require that the input determine the output uniquely. What follows is an example of an input-output system.

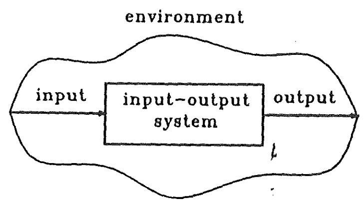


Figure 1.3 Input-output system.

1.2.2. Example: Resistive element connected to a voltage source. The resistive circuit element of Example 1.2.1 may be connected to a voltage source that produces a voltage v as in Fig. 1.4. The system we obtain still has the voltage v and current i as associated signals, but it now is natural to designate the voltage v as the *input* signal and the current i as the *output* signal. For some input voltages v, though, there is no unique output current i. ∎

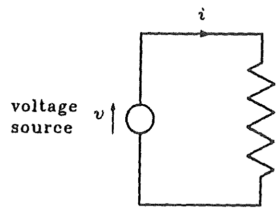


Figure 1.4 Resistive element connected to a voltage source.

Input-Output Mapping Systems

An input-output system whose input *uniquely* determines the output is called an *input-output mapping system*. The following example illustrates this notion.

1.2.3. Example: Parity bit generator. A parity bit generator as in Fig. 1.5 is a digital device that accepts words consisting of a fixed number of bits and complements this word with either a 0 or a 1 in such a way that the total number of ones is *even*. Thus, if the input is, say, an eight-bit word of the form $u = u_0u_1u_2u_3u_4u_5u_6u_7$, with u_k either 0 or 1 for $k = 0, 1, \cdots, 7$, the corresponding output is a nine-bit word $y = y_0y_1y_2y_3y_4y_5y_6y_7y_8$, where

$$y_k = \begin{cases} u_k & \text{for } k = 0, 1, \cdots, 7, \\ (u_0 + u_1 + \cdots + u_7) \bmod 2 & \text{for } k = 8. \end{cases}$$

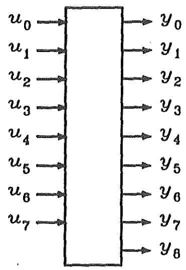


Figure 1.5 Parity bit generator.

If the input is, for instance, $u = 11001100$, the output is $y = 110011000$. This is an input-output mapping system, which maps every eight-bit byte into a nine-bit byte. ∎

Time Signals

In many instances we need deal with systems whose signals are *functions of time*. Such signals are called *time signals*. The collection of time instants on which the signal is defined is called its *time axis*. Time signals whose time axis is a *finite* or *countable* set of real numbers are called *discrete-time* signals. Time signals whose time axis is a finite or infinite *interval* of real numbers are called *continuous-time* signals. Figure 1.6 illustrates discrete-time and continuous-time signals.

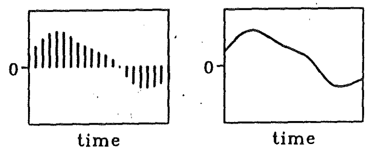


Figure 1.6 Time signals. Left: discrete-time signal. Right: continuous-time signal.

1.2.4. Example: Dow-Jones averages. The *Dow-Jones average* is a weighted average of the prices of a selected portfolio of stocks traded at the New York stock exchange, which has been daily computed and recorded since 1897. The sequence of Dow-Jones averages $z = (z(0), z(1), z(2), z(3), \cdots)$, with $z(n)$ denoting the Dow-Jones average on day n, is an example of a discrete-time signal. The time axis of this signal is the countable set of nonnegative integers $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. ∎

1.2.5. Example: Continuous-time signals. The outdoor temperature at any given place taken as a continuous function of time is a typical example of a continuous-time signal. The physical world abounds with continuous-time signals. ∎

Discrete-Time Systems

Systems associated with discrete-time signals are called *discrete-time systems*. An example of a discrete-time system that is used frequently in the rest of the book follows.

1.2.6. Example: Exponential smoother. In signal processing sometimes a procedure called *exponential smoothing* is used to remove unwanted fluctuations from observed time series, such as the Dow-Jones averages in Example 1.2.4. Let $u = (u(0), u(1), u(2), \cdots)$ be a discrete-time signal defined on the time axis $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. Then, as we observe the successive values of the time signal u we may form from these values another time signal y, which is a smoothed version of the signal u, according to

$$y(n + 1) = ay(n) + (1 - a)u(n + 1), \tag{1}$$

for $n = 0, 1, 2, \cdots$. Here, a is a constant such that $0 < a < 1$. This describes a discrete-time system. Equation (1) forms the *rule* of the system. To apply the rule, starting at time 0, we need specify an *initial value* $y(0)$. At each time $n + 1$, the output $y(n + 1)$ is formed as a weighted average of the new input $u(n + 1)$ at time $n + 1$ and the output $y(n)$ at the preceding time instant n. The closer the constant a is to 1, the more the preceding output value is weighted and the "smoother" the output is.

By repeated substitution it is easily found that

$$y(n) = a^n y(0) + (1 - a) \sum_{k=0}^{n-1} a^k u(n - k),$$

for $n = 1, 2, 3, \cdots$. This shows that $y(n)$ is a weighted sum of the present input, all past inputs back to $u(1)$, and the initial value $y(0)$. If $|a| < 1$, the effect of the initial value $y(0)$ asymptotically vanishes as n increases, and the weighting coefficients a^k become exponentially smaller the farther back the input values are in the past. This is why this scheme is called *exponential smoothing*. Figure 1.7 illustrates it.

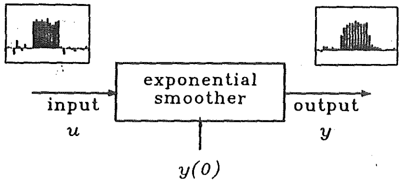


Figure 1.7 Exponential smoothing.

Because the output y is not uniquely determined by the input but also depends on the initial value $y(0)$, the exponential smoother is an input-output system but not an input-output mapping system. ■

Continuous-Time Systems

Continuous-time systems are systems that are associated with continuous-time signals. An example of a continuous-time system that will often return is the following RC network.

1.2.7. Example: RC network. The electrical circuit of Fig. 1.8 consists of a series connection of a voltage source, a resistor with resistance R, and a capacitor with capacitance C. The voltage u produced by the voltage source varies with time and constitutes the input to the system. The output y is the voltage across the capacitor. Both the input and output are continuous-time signals so that this is an example of a continuous-time system.

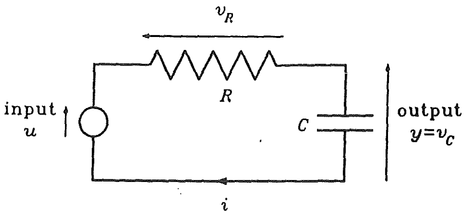


Figure 1.8 An RC network.

By Kirchhoff's voltage law, at each time t

$$u(t) = v_R(t) + v_C(t),$$

where v_R is the voltage across the resistor and v_C that across the capacitor. Denoting the current through the circuit as i, we have $v_R(t) = R\,i(t)$. Since, moreover, $i(t) = C\,dv_C(t)/dt$, it follows that

$$u(t) = RC\frac{dv_C(t)}{dt} + v_C(t).$$

Substitution of $v_C = y$ and division by RC leads to the differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}u(t).$$

Any input-output pair (u, y) need satisfy the differential equation, which forms the *rule* of the system.

For a given initial voltage $y(t_o) = v_C(t_o)$ and a given input voltage $u(t)$ for $t \geq t_o$, we may solve for $y(t)$, $t \geq t_o$, as

$$y(t) = e^{-(t-t_o)/RC}\,y(t_o) + \frac{1}{RC}\int_{t_o}^{t} e^{-(t-\tau)/RC}\,u(\tau)\,d\tau, \qquad t \geq t_o.$$

At each time t, the output $y(t)$ is a weighted sum of a term that is determined by the initial condition $y(t_o)$ and a term that is an exponentially weighted integral of past inputs back to the initial time t_o. The effect of the system is much analogous to that of the exponential smoother of Example 1.2.6. Figure 1.9 shows the system's response to the step signal

$$u(t) = \begin{cases} 0 & \text{for } 0 \leq t < a, \\ 1 & \text{for } t \geq a, \end{cases}$$

with $y(0) = 0$. The network turns the step into a smoothed version of a step given by

$$y(t) = \begin{cases} 0 & \text{for } 0 \leq t < a, \\ 1 - e^{-(t-a)/RC} & \text{for } t \geq a. \end{cases}$$ ■

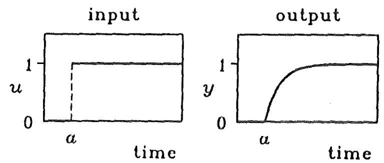


Figure 1.9 Response of the RC network. Left: step input. Right: output.

1.3 APPLICATIONS OF SIGNAL AND SYSTEM THEORY

Signal and system theory is applied in many fields of engineering and science. In this section we describe examples of practical engineering design problems whose solution requires the theory of signals and systems as developed in this book. The design problems are taken from the areas of *signal processing*, *communication engineering*, and *automatic control*.

Application to Signal Processing

A typical problem in the area of signal processing is the removal of unwanted noise from a signal. Examples are signals received over a telephone line or picked up from the read head of a tape recorder. The model that is frequently used is that the received signal u is given by

$$u = s + n,$$

where s is the transmitted signal and n is *noise* added to the signal. The problem is to design a *filter* as in Fig. 1.10, whose function is to remove or at least attenuate the noise. The input to the filter is the received signal u, and its output y is required to be reasonably similar to the transmitted signal s.

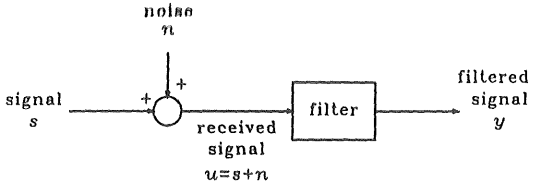


Figure 1.10 The filter design problem.

The exponential smoother of Example 1.2.6 may be viewed as such a filter, because it removes short-term irregularities from the input. The smoother is described by the equation

$$y(t + T) = ay(t) + (1 - a)u(t + T), \tag{1}$$

where we now let t take values on the time axis $\{t_o, t_o + T, t_o + 2T, \cdots\}$. The closer the constant a is to 1, the stronger is the smoothing effect. The stronger the smoothing effect, the more the undesired short-term fluctuations are suppressed. On the other hand, also part of the long-term changes we are actually interested in are smoothed over and eliminated. The choice of the constant a therefore is a crucial aspect of the design of the filter. To make a rational choice, a theory is needed that enables us to study the effect of the filter on both the signal and the noise.

By using modern digital circuitry it is very simple to implement the exponential smoother as a *digital filter*, which accepts a discrete-time signal as input and produces a filtered discrete-time signal as output. Because of their reliability and insensitivity to noise, digital filters are being used on an increasingly large scale in communication, audio and video equipment. Design methods for such filters are discussed in Chapter 9.

Application to Communication Engineering

Communication engineering deals with transferring information, such as audio and video signals and data streams, from one location to another. Since such signals do not propagate over long distances, it is necessary, if the locations are far apart, to

"mount" the signal on some *carrier* that may easily be transmitted over great distances.

One much used possibility is to employ high-frequency waves as carrier. At the point of origin the audio, video, or data signal, commonly called the *message* signal, is mounted on the carrier by a process called *modulation*. The modulated carrier propagates through the transmission medium (which may be space, cable, or another medium), and at the destination the message signal is retrieved from the carrier through a process called *demodulation*. Figure 1.11 illustrates such a transmission link.

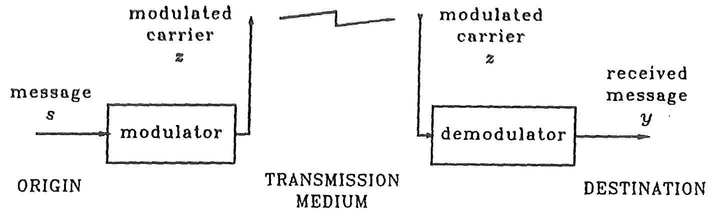


Figure 1.11 Long-distance radio transmission.

Let the continuous-time signal m be the message signal that is to be transmitted, and suppose that the continuous-time signal c is a high-frequency harmonic carrier with frequency f_c given by

$$c(t) = \cos(2\pi f_c t), \qquad -\infty < t < \infty.$$

Then *amplitude modulation* of the carrier c with the signal m results in the *amplitude modulated* signal z given by

$$z(t) = [m_o + m(t)] \cos(2\pi f_c t), \qquad -\infty < t < \infty.$$

The number m_o is a positive constant such that $m_o + m(t) \geq 0$ for all t. Figure 1.12 shows a modulated signal.

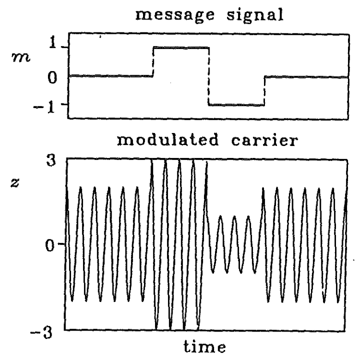


Figure 1.12 Amplitude modulation. Top: message signal. Bottom: modulated carrier.

An important subject in communication theory is the study of modulation schemes such as amplitude and frequency modulation. Chapter 10 presents an introduction to the theory of modulation.

Application to Automatic Control

Control engineering concerns itself with the design of automatic controllers, whose purpose is to govern the dynamic behavior of a given system. By way of example we consider an *automatic cruise controller* in a car. This controller automatically adjusts the engine throttle in such a way that the speed of the car maintains a *reference speed* set by the driver.

The cruise controller is arranged as in the block diagram of Fig. 1.13. The speed v of the car is continuously measured and converted to an electrical voltage. This voltage is compared to a voltage that represents the reference cruise speed v_r that has been set. The error signal $v_r - v$ is transmitted, again in the form of an electrical signal, to an electromechanical device that controls the throttle opening. The operation of this device is such that if the error $v_r - v$ is positive (i.e., the desired speed is *higher* than the actual speed), the throttle opening keeps increasing so that the speed of the car also increases until the speed reaches the desired value v_r. On the other hand, if the reference speed is lower than the actual speed, the throttle opening keeps decreasing. The configuration of Fig. 1.13 is called a *feedback system*, because the output is returned to the input.

The selection of a suitable control mechanism is a central issue in control system design. The theory of feedback and some aspects of control system design are discussed in Chapter 11.

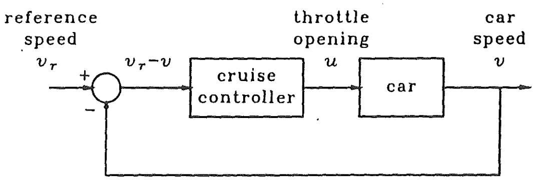


Figure 1.13 Automatic cruise control system.

2

An Introduction to Signals

2.1 INTRODUCTION

In the context of this book, a *signal* is a phenomenon that represents information. Since any signal always is one of a collection of several of many possible signals, signals may mathematically be represented as elements of a set, called the *signal set*. In this chapter we introduce a variety of signal sets. Moreover, operations on and among signals are defined and discussed.

In Section 2.2 signals are defined. We are primarily interested in *time signals*. Section 2.3 deals with elementary operations *on* and *among* signals, such as *signal range* and *signal axis transformation, sampling, interpolation*, and *pointwise addition, multiplication*, and *division*.

In Section 2.4, *signal spaces* are introduced, along with the notions of *norm* and *inner product*. Various important signal spaces, such as spaces of *bounded amplitude, bounded action*, and *bounded energy* signals, are defined.

Section 2.5 is devoted to *generalized* signals. The most prominent example of a generalized signal is the δ-function, which may be viewed as a signal of zero duration, infinite height, and unit area.

In Supplement A at the end of the book various elementary facts about *complex numbers, sets,* and *maps* are reviewed. This material is assumed to be familiar throughout the book.

2.2 SIGNALS

In this section we introduce various kinds of signals, such as discrete- and continuous-time signals, finite- and infinite-time signals, and periodic and harmonic signals.

The signals we are interested in are *functions* of a variable that often is *time*. The *domain* of a signal is a subset $\mathbb{T}$ of the real line and is called the *signal axis*. The signal may take values in any set A, called the *signal range*. Figure 2.1 illustrates the idea. The formal definition of a signal is as follows.

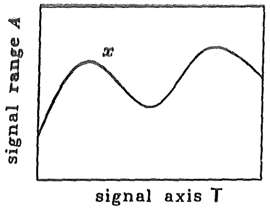


Figure 2.1 A signal is a function $\mathbb{T} \to A$.

2.2.1. Definition: Signals. Let A be a set, and suppose that $\mathbb{T}$ is a subset of the reals $\mathbb{R}$. Then, any function $x : \mathbb{T} \to A$ is called a *signal* with *signal axis* $\mathbb{T}$ and *signal range* A. ■

The set of all signals with signal axis $\mathbb{T}$ and signal range A thus is the set of all functions from $\mathbb{T}$ to A. This signal set is denoted by $\{x : \mathbb{T} \to A\}$, or by the *power set* notation $A^{\mathbb{T}}$ (see Supplement A).

If the signal axis has the interpretation of time, the signal is called a *time signal*, and the signal axis is called *time axis*. We also meet other signals, in particular *frequency signals*.

2.2.2. Examples: Signals.

(a) *Dow-Jones averages*. The Dow-Jones averages of Example 1.2.4 form a time signal with time axis $\mathbb{T} = \mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and signal range $A = \mathbb{R}_+$, the set of nonnegative real numbers.

(b) *Bit stream*. A semi-infinite bit stream such as $1010011 \cdots$ is a signal with signal axis $\mathbb{T} = \mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and signal range $A = \{0, 1\}$.

(c) *Electrical signal*. The voltage across the capacitor of the RC network of Example 1.2.7 is a time signal with axis $\mathbb{T} = [t_o, \infty)$ and signal range $A = \mathbb{R}$. ■

Discrete- and Continuous-Time Signals

Time signals may either be *discrete-* or *continuous-time* signals:

2.2.3. Definition: Discrete and continuous time axes; discrete- and continuous-time signals.

(a) The time axis $\mathbb{T} \subset \mathbb{R}$ is *discrete* if it consists of a *finite* or *countable* set of time instants. A time signal whose time axis is discrete is called a *discrete-time signal*.

(b) The time axis $\mathbb{T} \subset \mathbb{R}$ is *continuous* if it consists of an interval of $\mathbb{R}$, possibly extending to $-\infty$ or $+\infty$ or to both. A time signal whose time axis is continuous is called a *continuous-time* signal. ■

In this text, the signal range A usually is the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$. If $A = \mathbb{R}$, the signal is said to be *real-valued*, while, if $A = \mathbb{C}$, it is called *complex-valued*.

2.2.4. Examples: Discrete- and continuous-time signals.

(a) *A real-valued discrete-time signal*. The time axis $\mathbb{T} = \{0, 1, 2, 3, 4\}$ is discrete. The signal $x \in \mathbb{R}^{\mathbb{T}}$ shown in Fig. 2.2, defined by

$$x(n) = n + 1, \qquad n \in \{0, 1, 2, 3, 4\},$$

is an example of a real-valued discrete-time signal.

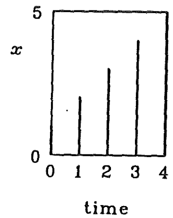


Figure 2.2 A real-valued discrete-time signal.

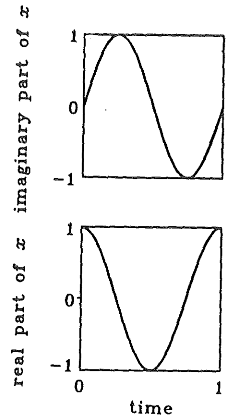


Figure 2.3 A complex-valued continuous-time signal.

(b) *A complex-valued continuous-time signal*. The set $\mathbb{T} = [0, 1]$ is a continuous time axis, and the signal $x \in \mathbb{C}^{\mathbb{T}}$ defined by

$$x(t) = e^{j2\pi t} = \cos(2\pi t) + j\sin(2\pi t), \qquad t \in \mathbb{T},$$

as shown in Fig. 2.3, is a complex-valued continuous-time signal. ■

Time Sequences and Sampled Signals

Suppose that x is a discrete-time signal defined on the time axis

$$\mathbb{T} = \{t_0, t_1, t_2, \cdots\},$$

with $t_0 < t_1 < t_2 < \cdots$. We may always list the consecutive values

$$x(t_0),\ x(t_1),\ x(t_2),\ \cdots \tag{1}$$

of the signal as a *sequence*. Thus, any discrete-time signal may either be seen as a *map* from a time axis $\mathbb{T}$ to its signal range A or, alternatively, as an *ordered sequence* of values in A. This ordered sequence may be redefined on a time axis consisting of *integers* so that the sequence (1) is rewritten as

$$x(0),\ x(1),\ x(2),\ \cdots.$$

In the sequel, we refer to discrete-time signals defined on time axes that consist of consecutive integers, such as the set $\mathbb{N}$ of all natural numbers, the set $\mathbb{Z}_+$ of all nonnegative integers, and the set $\mathbb{Z}$ of all integers, as *time sequences*.

Often, though not always, a discrete-time signal is obtained by observing a phenomenon that takes place on a continuous time axis at a sequence of discrete time instants $t_0, t_1, t_2, \cdots$, called the *sampling times*. The discrete-time signal obtained in this way is referred to as a *sampled signal*. If the sampling times are *uniformly spaced* (i.e., $t_i = iT$ with i ranging over a set of consecutive integers and $T > 0$), we say that the signal is *uniformly sampled* with *sampling interval T*. The number of samples per unit of time $1/T$ is called the *sampling rate* of the sampled signal.

2.2.5. Example: Sampled signal and time sequence. Consider the continuous time axis $[0, \infty)$ and the continuous-time signal x defined on this axis by

$$x(t) = \frac{1}{1 + \dfrac{t}{T}}, \qquad t \in [0, \infty),$$

with $T > 0$, as illustrated in Fig. 2.4(a).

Observing the signal x on the discrete time axis $\{0, T, 2T, \cdots\}$ results in the *uniformly sampled signal* x^* defined by

$$x^*(t) = \frac{1}{1 + \dfrac{t}{T}}, \qquad t \in \{0, T, 2T, \cdots\},$$

as shown in Fig. 2.4(b).

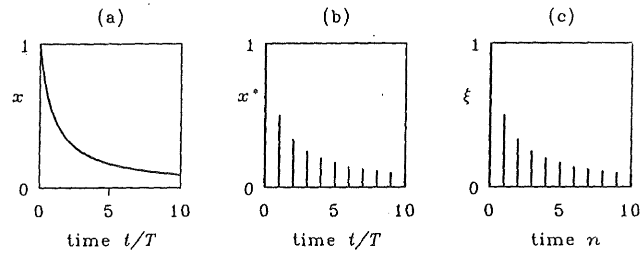


Figure 2.4 Sampled signal and time sequence. Left: a continuous-time signal. Middle: the sampled signal. Right: the resulting time sequence.

The consecutive values taken by the sampled signal x^* form the *time sequence* ξ defined by

$$\xi(n) = x^*(nT) = x(nT) = \frac{1}{1+n}, \qquad n = 0, 1, 2, \cdots,$$

as in Fig. 2.4(c). The time sequence ξ is defined on the time axis $\mathbb{Z}_+$. Note that the sampled signal x^* and the time sequence ξ consecutively assume the *same* values but have *different* time axes. ∎

In the sequel, important results that relate to discrete-time signals are first formulated for *time sequences*. Nearly always, the corresponding results for *uniformly sampled signals* are reviewed in a form that emphasizes the parallels between the discrete- and the continuous-time case. The sampled results may be specialized to sequences by simply setting the sampling interval T equal to 1.

Finite-Time, Semi-Infinite-Time, and Infinite-Time Signals

A time signal may be *finite-time*, *semi-infinite-time*, or *infinite-time*, depending on whether its time axis is finite, semi-infinite, or infinite.

2.2.6. Definition: Finite, semi-infinite and infinite time axes.

(a) *Finite time axis.* If a time axis, whether discrete or continuous, is contained in a finite interval, it is called a *finite* time axis.

(b) *Semi-infinite time axis.* If a time axis is bounded from the left, it is called a *right semi-infinite* time axis, while, if it is bounded from the right, it is said to be *left semi-infinite*.

(c) *Infinite time axis.* A time axis that is neither bounded from the left nor from the right is called an *infinite* time axis. ∎

The most common discrete time axes in this text follow. For time sequences we have the *infinite* time axis $\mathbb{Z}$ consisting of all integers, the *right semi-infinite* time axis $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ consisting of all nonnegative integers, and the *finite* time axis

$$\underline{N} := \{0, 1, 2, \cdots, N-1\},$$

with N any natural number. Note that the notation $\underline{N}$ may be used numerically so that $\underline{4}$ denotes the set $\{0, 1, 2, 3\}$.

For uniformly sampled signals with sampling interval T we correspondingly have the infinite, (right) semi-infinite, and finite time axes

$$\mathbb{Z}(T) = \{\cdots, -T, 0, T, 2T, \cdots\},$$
$$\mathbb{Z}_+(T) = \{0, T, 2T, \cdots\},$$
$$\underline{N}(T) := \{0, T, 2T, \cdots, (N-1)T\},$$

respectively, where the latter is defined for any natural number N.

The usual continuous time axes are the infinite time axis $\mathbb{R}$, the (right) semi-infinite time axis $\mathbb{R}_+$, and finite time axes of the form $[a, b]$, with $-\infty < a < b < \infty$.

In the sequel we occasionally use the following notations for sets of time signals defined on different time axes:

ℓ	all complex-valued time sequences with the infinite time axis $\mathbb{Z}$,
ℓ_+	all complex-valued time sequences with the semi-infinite time axis $\mathbb{Z}_+$,
$\ell_{\underline{N}}$	all complex-valued time sequences with the finite time axis $\underline{N}$,
$\ell(T)$	all complex-valued uniformly sampled signals with the infinite time axis $\mathbb{Z}(T)$,
$\ell_+(T)$	all complex-valued uniformly sampled signals with the semi-infinite time axis $\mathbb{Z}_+(T)$,
$\ell_{\underline{N}}(T)$	all complex-valued uniformly sampled signals with the finite time axis $\underline{N}(T)$,
$\mathcal{L}$	all complex-valued continuous-time signals with the infinite time axis $\mathbb{R}$,
$\mathcal{L}_+$	all complex-valued continuous-time signals with the semi-infinite time axis $\mathbb{R}_+$,
$\mathcal{L}[a, b]$	all complex-valued continuous-time signals with the finite time axis $[a, b]$.

Some Well-Known Signals

In what follows we give examples of some familiar time signals.

2.2.7. Example: Some well-known signals.

(a) *The unit pulse.* The unit pulse of Fig. 2.5 is the time sequence Δ defined by

$$\Delta(n) = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad n \in \mathbb{T},$$

where $\mathbb{T}$ may be any of the discrete time axes $\underline{N}$, $\mathbb{Z}_+$, or $\mathbb{Z}$. Correspondingly, the unit pulse belongs to the signal set $\ell_{\underline{N}}$, ℓ_+, or ℓ.

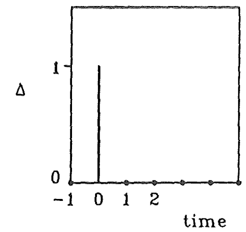


Figure 2.5 The unit pulse Δ.

(b) *Rectangular and triangular pulses.* Well-known continuous-time signals are the rectangular and triangular pulse, defined as

$$\operatorname{rect}(t) = \begin{cases} 1 & \text{for } -\tfrac{1}{2} \le t < \tfrac{1}{2}, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{R},$$

$$\operatorname{trian}(t) = \begin{cases} 1 - |t| & \text{for } |t| < 1, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{R}.$$

The two signals are sketched in Fig. 2.6. They both belong to the signal set $\mathcal{L}$.

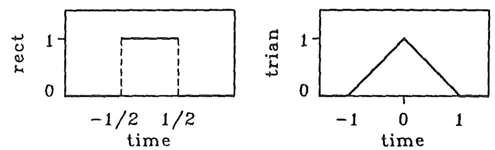


Figure 2.6 Left: rectangular pulse. Right: triangular pulse.

(c) *Unit step and ramp signal.* Two further well-known signals, which exist both in discrete- and continuous-time form, are the unit step $\mathbb{1}$ and the ramp signal, defined as

$$\mathbb{1}(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \ge 0, \end{cases} \qquad t \in \mathbb{T},$$

$$\operatorname{ramp}(t) = \begin{cases} 0 & \text{for } t < 0, \\ t & \text{for } t \ge 0, \end{cases} \qquad t \in \mathbb{T},$$

where $\mathbb{T}$ is any of the infinite time axes $\mathbb{Z}$, $\mathbb{Z}(T)$, or $\mathbb{R}$. The signals are displayed in Fig. 2.7. Depending on the time axis, the signals belong to ℓ, $\ell(T)$, or $\mathcal{L}$. ∎

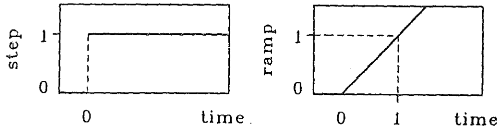


Figure 2.7 Left: unit step. Right: ramp signal.

Periodic Signals

A *periodic signal* is a signal that repeats itself indefinitely. The length of time after which the signal starts repeating itself is called its *period*. The formal definition goes as follows.

2.2.8. Definition: Periodic time signal.
(a) *Periodic signal.* A signal $x \in A^{\mathbb{T}}$ with signal range A and infinite axis $\mathbb{T} = \mathbb{Z}$, $\mathbb{Z}(T)$ or $\mathbb{R}$ is *periodic* if there exists a $P \in \mathbb{T}$ with $P > 0$ such that

$$x(t + P) = x(t) \qquad \text{for all} \quad t \in \mathbb{T}.$$

(b) *Period of a periodic signal.* The real number P is called the *period* of a periodic signal x if it is the smallest positive number such that

$$x(t + P) = x(t) \qquad \text{for all} \quad t \in \mathbb{T}. \qquad \blacksquare$$

Examples of a discrete- and a continuous-time periodic signal are given in Fig. 2.8. The reason that in part (b) of the definition we insist on the *smallest* P is that if $x(t) = x(t + P)$ for all $t \in \mathbb{T}$, it follows that $x(t) = x(t + qP)$ for all $t \in \mathbb{T}$ with q any natural number. The definition ensures that the period P of the signal is uniquely determined. Note that if x is a sampled discrete-time periodic signal, its period P necessarily is an integral multiple of the sampling interval T (i.e., $P = pT$, with p a natural number).

The inverse $1/P$ of the period of a periodic signal is the number of times per unit of time that the signal repeats itself and is called the *repetition rate* of the signal.

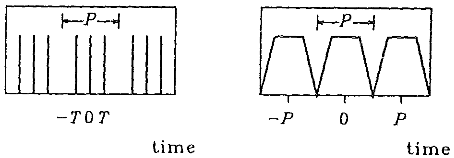


Figure 2.8 Period signals. Left: discrete-time. Right: continuous-time.

Harmonic Signals

The most conspicuous examples of periodic signals are *harmonic* signals, although not *all* harmonic signals are periodic.

Harmonic signals exist both in the discrete- and the continuous-time case. For any real number f, the (complex) harmonic signal η_f with *frequency* f is defined by

$$\eta_f(t) = e^{j2\pi ft}, \qquad t \in \mathbb{T},$$

where $\mathbb{T}$ is the time axis. The number

$$\omega := 2\pi f$$

is called the *angular frequency* of the harmonic. Also, signals of the form

$$x(t) = a\, e^{j2\pi ft}, \qquad t \in \mathbb{T},$$

with a any complex constant, are said to be harmonic. The constant a is called the *complex amplitude* of the harmonic signal. We may write the complex number a in polar form as

$$a = \alpha\, e^{j\phi},$$

with the magnitude $\alpha = |a| > 0$ and the angle ϕ both real numbers. Then,

$$\begin{aligned} x(t) &= a\, e^{j2\pi ft} = \alpha\, e^{j\phi}\, e^{j2\pi ft} = \alpha\, e^{j(2\pi ft + \phi)} \\ &= \alpha \cos(2\pi ft + \phi) + j\alpha \sin(2\pi ft + \phi), \qquad t \in \mathbb{T}. \end{aligned}$$

The real-valued signal c given by

$$\begin{aligned} c(t) &= \mathrm{Re}(x(t)) \\ &= \alpha \cos(2\pi ft + \phi), \qquad t \in \mathbb{T}, \end{aligned}$$

is said to be a *real harmonic* signal, with frequency f, *amplitude* α, and *phase* ϕ. The complex number $a = \alpha e^{j\phi}$ is called the *phasor* of the real harmonic signal c.

We next discuss the periodicity properties of harmonic signals. First, consider the continuous-time case, with $\mathbb{T} = \mathbb{R}$. Then, for $f \neq 0$, the harmonic signal η_f is periodic with period $P = 1/|f|$. The reason is that $P = 1/|f|$ is the smallest positive value of P such that $e^{j2\pi fP} = 1$. We summarize as follows.

2.2.9. Summary: Periodicity of continuous-time harmonics. The continuous-time harmonic η_f, defined on the time axis $\mathbb{R}$, with frequency $f \neq 0$, is periodic with period $P = 1/|f|$. Its repetition rate, hence, is $|f|$. $\blacksquare$

For $f = 0$, the harmonic signal η_f reduces to $\eta_0(t) = 1$ for all $t \in \mathbb{R}$, which is trivially periodic.

In the discrete-time case, things are a little more complicated. First of all, there are fewer discrete-time harmonic signals than it would seem. The reason is that, as Fig. 2.9 shows, sampling two continuous-time harmonics with *different* frequencies may result in the *same* discrete-time signal. This phenomenon is called *aliasing*.

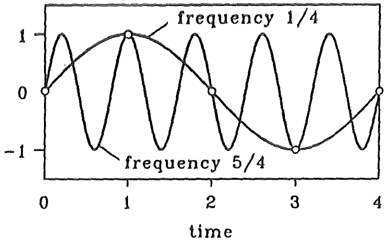


Figure 2.9 Aliasing: on the time axis $\mathbb{Z}$ the real harmonics $\sin(2\pi f_1 t)$ and $\sin(2\pi f_2 t)$ with $f_1 = 1/4$ and $f_2 = 5/4$ coincide.

2.2.10. Summary: Aliasing. On the time axis $\mathbb{Z}(T)$, the harmonic signals η_f and $\eta_{f+k/T}$ are identical for any $k \in \mathbb{Z}$. ∎

2.2.11. Proof. For $t \in \mathbb{Z}(T)$ it follows that $\eta_{f+k/T}(t) = e^{j2\pi(f+k/T)t} = e^{j2\pi ft} \cdot e^{j2\pi kt/T} = e^{j2\pi ft} = \eta_f(t)$, since kt/T is an integer for all $t \in \mathbb{Z}(T)$. ∎

As a result, discrete-time harmonics whose frequencies differ by an integral multiple of the sampling rate $1/T$ cannot be distinguished.

One consequence of aliasing is that when studying discrete-time harmonic signals on the time axis $\mathbb{Z}(T)$ we may as well restrict the range of frequencies that are considered to the interval $[0, 1/T)$. Sometimes it is useful to take the interval $[-1/2T, 1/2T)$.

A further fact to bear in mind is that by no means is every discrete-time harmonic signal periodic. The reason is that the sampling interval and the period of a continuous-time harmonic signal that is sampled are not necessarily commensurable.

2.2.12. Summary: Periodicity of discrete-time harmonic signals. The discrete-time harmonic signal η_f defined on the time axis $\mathbb{Z}(T)$ is periodic if and only if the frequency f is a *rational* multiple of $1/T$. If $|f| = p/qT$, with p and q coprime natural numbers, the harmonic has period qT. ∎

2.2.13. Proof. As observed before, the period P of any periodic signal on the time axis $\mathbb{Z}(T)$ is of the form $P = qT$, with q a natural number. The harmonic signal η_f, hence, is periodic on the time axis $\mathbb{Z}(T)$ if and only if $\eta_f(t) = \eta_f(t + qT)$ for all $t \in \mathbb{Z}(T)$ for some natural number q, that is, when $e^{j2\pi ft} = e^{j2\pi f(t+qT)} = e^{j2\pi ft} \cdot e^{j2\pi fqT}$ for all $t \in \mathbb{Z}(T)$. This holds if and only if $|fqT|$ is a natural number, say p, which is equivalent to the condition that $|f| = p/qT$. This in turn amounts to the statement that $|f|$ is a rational multiple of $1/T$. If $|f|$ is a rational multiple of $1/T$, the smallest q such that $|f| = p/qT$ is obtained by taking p and q coprime. ∎

It is easy to see that, although on the time axis $\mathbb{Z}(T)$ the harmonic η_f in general does not repeat itself exactly, it repeats itself *approximately* at a rate $|f|$ if $-1/2T \leq$

$f < 1/2T$. One way of establishing this is by noting that $|f|$ is the average number of maxima and minima of the real and imaginary parts of η_f per unit of time (see Problem 2.6.9). Figure 2.10 illustrates this. If f does not lie between $-1/2T$ and $1/2T$, let $f' = f - k/T$ with $k \in \mathbb{Z}$ such that f' lies in the interval $[-1/2T, 1/2T)$. By 2.2.10, on the time axis $\mathbb{Z}(T)$ the harmonic η_f is indistinguishable from the harmonic $\eta_{f'}$. As a result, the average repetition rate of η_f is $|f'|$.

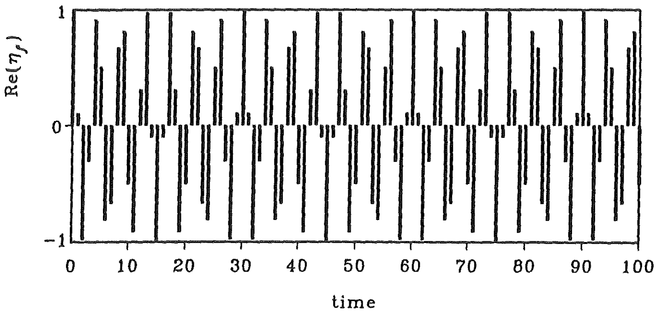


Figure 2.10 Real part of the harmonic η_f on the time axis $\mathbb{Z}$ with $f = 7/30$. The signal repeats itself exactly every 30 sampling intervals and approximately every $30/7 \approx 4.3$ sampling intervals.

2.3 ELEMENTARY OPERATIONS ON AND AMONG TIME SIGNALS

In this section we discuss various operations that *modify* signals. A distinction is made between *unary* and *binary* operations. Unary operations involve a *single* time signal, while binary operations require *two* signals. Important unary operations are *range* and *domain transformation,* which modify the signal range and the signal axis, respectively, and *sampling* and *interpolation,* which convert continuous-time to discrete-time signals and vice-versa. The binary operations we consider are (pointwise) *addition, subtraction, multiplication,* and *division.*

Signal Range Transformation

A signal range transformation changes the *range* of a time signal.

2.3.1. Definition: Signal range transformation. Let $\mathbb{T}$ be a signal axis, A_{old} a signal range and ρ: $A_{\text{old}} \to A_{\text{new}}$ a map from A_{old} to another signal range A_{new}. Then, *range transformation* of the signal $x_{\text{old}} \in A_{\text{old}}^{\mathbb{T}}$ under ρ results in the signal $x_{\text{new}} \in A_{\text{new}}^{\mathbb{T}}$ defined by

$$x_{\text{new}}(t) = \rho(x_{\text{old}}(t)), \qquad t \in \mathbb{T}.$$

Note that range transformation is actually map composition from the *left* because $x_{\text{new}} = \rho \circ x_{\text{old}}$. (See Supplement A for the definition of map composition.) Range transformation is a *pointwise* operation in the sense that $x_{\text{new}}(t)$ for any $t \in \mathbb{T}$ is fully determined by $x_{\text{old}}(t)$.

2.3.2. Example: Full-wave rectification. The amplitude transformation ρ: $\mathbb{R} \to \mathbb{R}_+$ defined by $\rho(x) = |x|$ is called *full-wave rectification*. Figure 2.11 illustrates what it does to a real harmonic signal.

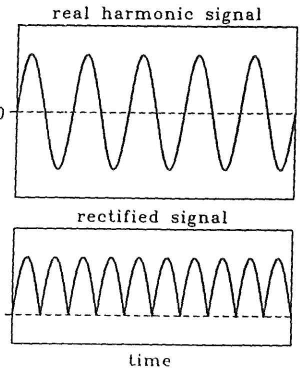


Figure 2.11 Full-wave rectification. Top: before rectification. Bottom: after. ∎

Quantization

An important range transformation is *quantization*. This transformation is needed when signals are processed by a digital computer or other digital equipment, because computers can only handle signals that have a *finite* range.

2.3.3. Definition: Quantization. Any range transformation such that A_{new} is a finite set is called *quantization*. ∎

The following example illustrates *uniform* quantization.

2.3.4. Example: Uniform quantization. Suppose that $A_{\text{old}} = \mathbb{R}$, and let us range transform to the finite signal range $A_{\text{new}} = \underline{N}(H) = \{0, H, 2H, \cdots, (N-1)H\}$ by ρ: $\mathbb{R} \to \underline{N}(H)$, where

$$\rho(x) = \begin{cases} 0 & \text{for } x < 0, \\ H \operatorname{int}(x/H) & \text{for } 0 \le x < (N-1)H, \\ (N-1)H & \text{for } x \ge (N-1)H. \end{cases}$$

Here, int is the *entier* function defined such that int(x) is the greatest integer less than or equal to the real number x. N is a natural number and H a positive number

called the *quantization interval*. The graph of ρ is shown in Fig. 2.12. Quantiz according to this function implies that signal values below 0 are set equal to 0, si values between 0 and $(N-1)H$ are rounded down to the nearest value in th $\underline{N}(H) = \{0, H, 2H, \cdots, (N-1)H\}$, while values above $(N-1)H$ are set e to $(N-1)H$. We refer to this transformation as *uniform quantization*.

Figure 2.13 illustrates the effect of uniform quantization with $N = 8$ le and quantization range $\{0, 1/(N-1), 2/(N-1), \cdots, 1\}$ on the discrete-t signal x defined by $x(t) = \frac{1}{2}[1 - \cos(2\pi t)]$, $t \in \mathbb{Z}(T)$, where $T = 1/50$.

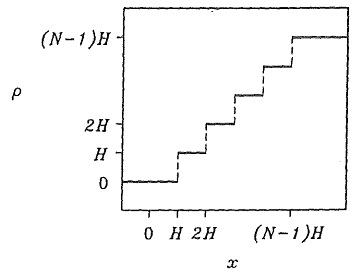


Figure 2.12 Range transformation for uniform quantization.

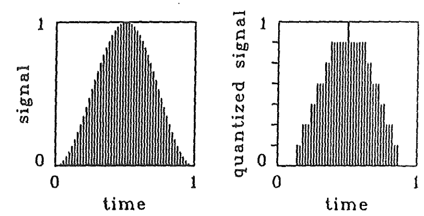


Figure 2.13 Uniform quantization. Left: before quantization. Right: after.

Signal Axis Transformation

We continue with defining *signal axis* or *signal domain* transformations.

2.3.5. Definition: Signal axis transformation. Let A be a signal range, $\mathbb{T}_{\text{old}}$ a signal axis, and τ: $\mathbb{T}_{\text{old}} \to \mathbb{T}_{\text{new}}$ a bijective map from the signal axis $\mathbb{T}_{\text{old}}$ to another signal axis $\mathbb{T}_{\text{new}}$, with inverse τ^{-1}: $\mathbb{T}_{\text{new}} \to \mathbb{T}_{\text{old}}$. Then, *signal axis transformation* of the signal $x_{\text{old}} \in A^{\mathbb{T}_{\text{old}}}$ under τ results in the signal $x_{\text{new}} \in A^{\mathbb{T}_{\text{new}}}$ defined on the signal axis $\mathbb{T}_{\text{new}}$, given by

$$x_{\text{new}}(t) = x_{\text{old}}(\tau^{-1}(t)) \qquad \text{for all} \quad t \in \mathbb{T}_{\text{new}}.$$ ∎

For the definition of a bijective map, see Supplement A. Signal axis transformation is map composition from the *right* by τ^{-1}, because $x_{\text{new}} = x_{\text{old}} \circ \tau^{-1}$. The diagram of Fig. 2.14 illustrates this.

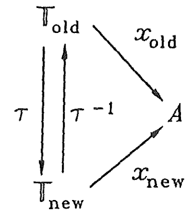


Figure 2.14 Signal axis transformation.

2.3.6. Example: Time expansion, time compression, and time reversal. Time *expansion*, time *compression*, and time *reversal* result by application of a time axis transformation of the form

$$\tau(t) = t/\alpha, \qquad t \in \mathbb{T}_{\text{old}},$$
$$\tau^{-1}(t) = \alpha t, \qquad t \in \mathbb{T}_{\text{new}},$$

so that

$$x_{\text{new}}(t) = x_{\text{old}}(\alpha t), \qquad t \in \mathbb{T}_{\text{new}},$$

Here, α is a nonzero real constant. If $\alpha > 1$, time is *compressed;* if $0 < \alpha < 1$, time is *expanded;* while, if $\alpha = -1$, time is *reversed*. Collectively, these transformations are referred to as *time scaling*.

Let x_{old} be the tooth-shaped pulse defined by

$$x_{\text{old}}(t) = \begin{cases} t/b & \text{for } 0 \le t < b, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{R},$$

where b is a positive real constant. This signal is defined on the time axis $\mathbb{T}_{\text{old}} = \mathbb{R}$. Time compression, expansion, or reversal result in a new time axis $\mathbb{T}_{\text{new}} = \mathbb{R}$ that coincides with the old time axis. In Figs. 2.15(c)–(e), plots are given of the time compressed signal, the time expanded signal, and the time reversed signal.

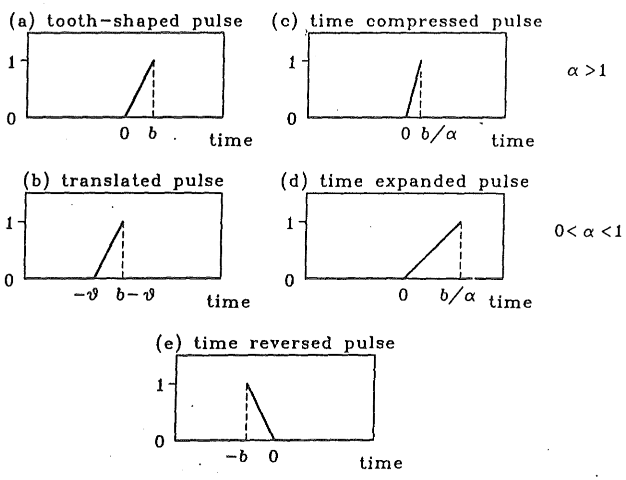


Figure 2.15 Time transformations of the tooth-shaped pulse. ∎

2.3.7. Example: Time translation. Time translation consists of the application of a time axis transformation of the form

$$\tau(t) = t - \theta, \qquad t \in \mathbb{T}_{\text{old}},$$
$$\tau^{-1}(t) = t + \theta, \qquad t \in \mathbb{T}_{\text{new}},$$

where θ is a real constant. Time translation of x_{old} results in the translated signal

$$x_{\text{new}}(t) = x_{\text{old}}(t + \theta), \qquad t \in \mathbb{T}_{\text{new}}.$$

Figure 2.15(b) shows the effect of time translation on the tooth-shaped pulse of Example 2.3.6. The time axis again remains unchanged. ∎

Sampling and Interpolation

Many physical signals, such as electrical voltages produced by a sound or image recording instrument or a measuring device, are essentially continuous-time signals. Computers and related devices operate on a discrete time axis. Continuous-time signals that are to be processed by such devices therefore first need be converted to discrete-time signals. One way of doing this is *sampling*.

2.3.8. Definition: Sampling. Let A be any signal range, $\mathbb{T}_{\text{con}}$ a continuous time axis, and $\mathbb{T}_{\text{dis}} \subset \mathbb{T}_{\text{con}}$ a discrete time axis. Then, *sampling* the continuous time signal $x \in A^{\mathbb{T}_{\text{con}}}$ on the discrete time axis $\mathbb{T}_{\text{dis}}$ results in the sampled signal $x^* \in A^{\mathbb{T}_{\text{dis}}}$ defined by

$$x^*(t) = x(t) \qquad \text{for all} \quad t \in \mathbb{T}_{\text{dis}}. \qquad ∎$$

A device that performs the sampling operation is called a *sampler* and is schematically represented as in Fig. 2.16.

Figure 2.16 Sampler.

2.3.9. Example: Sampled real harmonic. Let the continuous-time signal x, given by $x(t) = \frac{1}{2}[1 - \cos(2\pi t)]$, $t \in \mathbb{R}$, be sampled on the uniformly spaced discrete time axis $\mathbb{Z}(T)$. This results in the sampled signal x^* given by

$$x^*(t) = \tfrac{1}{2}[1 - \cos(2\pi t)], \qquad t \in \mathbb{Z}(T).$$

Figure 2.17 depicts the two signals on the interval $[0, 1]$ for $T = 1/50$.

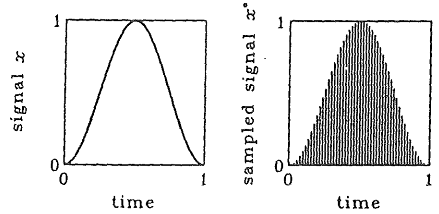


Figure 2.17 Sampling. Left: a continuous-time signal. Right: its sampled version. ∎

The converse problem of sampling presents itself when a discrete-time device, such as a computer, produces signals that need drive a physical instrument requiring a continuous-time signal as input. Suppose that a discrete-time signal x^* is defined on the discrete time axis $\mathbb{T}_{\text{dis}}$ and that we wish to construct from x^* a continuous-time signal x defined on the continuous time axis $\mathbb{T}_{\text{con}} \supset \mathbb{T}_{\text{dis}}$. There obviously are many ways to do this. We introduce a particular class of conversions from discrete-time to continuous-time signals, for which we reserve the term *interpolation*. This type of conversion has the property that the continuous-time signal x agrees with the discrete-time signal x^* *at* the sampling times.

2.3.10. Definition: Interpolation. Suppose that A is a signal range, $\mathbb{T}_{\text{dis}}$ a discrete time axis, and $\mathbb{T}_{\text{con}} \supset \mathbb{T}_{\text{dis}}$ a continuous time axis. Let $x^* \in A^{\mathbb{T}_{\text{dis}}}$ be a given discrete-time signal. Then, any continuous-time signal $x \in A^{\mathbb{T}_{\text{con}}}$ is called an *interpolation* of x^* on $\mathbb{T}_{\text{con}}$ if

$$x(t) = x^*(t) \qquad \text{for all} \quad t \in \mathbb{T}_{\text{dis}}. \qquad ∎$$

Another way of saying that x is an interpolation of x^* is the statement that sampling the continuous-time signal x generated by interpolating the discrete-time signal x^* on $\mathbb{T}_{\text{dis}}$ reproduces the discrete-time signal x^*.

Clearly, there is no unique interpolation for a given discrete-time signal x^*. Suppose that x^* is defined on the uniformly sampled discrete time axis $\mathbb{Z}(T)$. Then, one possible interpolation method is *step interpolation* as illustrated in Fig. 2.18(a).

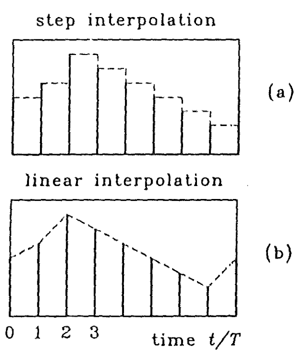


Figure 2.18 Interpolation. Top: step interpolation. Bottom: linear interpolation.

Given x^*, the step interpolated signal x is given by

$$x(t) = x^*(kT), \qquad kT \le t < (k+1)T, \qquad k \in \mathbb{Z}.$$

Another interpolation method is *linear interpolation*, as in Fig. 2.18(b). Given x the interpolated signal x is

$$x(t) = \left(1 - \frac{t - kT}{T}\right) x^*(kT) + \frac{t - kT}{T} x^*((k+1)T),$$
$$kT \le t < (k+1)T, \qquad k \in \mathbb{Z}$$

Step and linear interpolation are examples of interpolation by *interpolating functions*. An interpolating function is any function $i\colon \mathbb{R} \to \mathbb{C}$ such that

$$i(t) = \begin{cases} 1 & \text{for } t = 0, \\ 0 & \text{for } t = n, \quad \text{where } n \ne 0, \quad n \in \mathbb{Z}, \end{cases} \qquad t \in \mathbb{R}.$$

If $x^* \in \ell(T)$ is a discrete-time signal defined on the time axis $\mathbb{Z}(T)$, and i an interpolating function, the continuous-time signal x given by

$$x(t) = \sum_{n \in \mathbb{Z}} x^*(nT)\, i\left(\frac{t - nT}{T}\right), \qquad t \in \mathbb{R},$$

is an interpolation of x^*. The reason is that by setting $t = kT$, with k an integer, it follows that

$$x(kT) = \sum_{n \in \mathbb{Z}} x^*(nT)\, i\left(\frac{kT - nT}{T}\right) = x^*(kT) \qquad \text{for} \quad k \in \mathbb{Z}.$$

Step interpolation is achieved with the *step interpolating function*

$$i_{\text{step}}(t) = \begin{cases} 1 & \text{for } 0 \le t < 1, \\ 0 & \text{otherwise}, \end{cases} \qquad t \in \mathbb{R},$$

while linear interpolation is obtained with the *linear interpolating function*

$$i_{\text{lin}}(t) = \begin{cases} 1 - |t| & \text{for } |t| < 1, \\ 0 & \text{otherwise}, \end{cases} \qquad t \in \mathbb{R}.$$

Another interpolation function is the *sinc interpolating function*

$$i_{\text{sinc}}(t) = \text{sinc}(\pi t).$$

Here, sinc: $\mathbb{R} \to \mathbb{R}$ is the function defined by

$$\operatorname{sinc}(t) = \begin{cases} \dfrac{\sin(t)}{t} & \text{for } t \neq 0, \\ 1 & \text{for } t = 0, \end{cases} \qquad t \in \mathbb{R}.$$

Graphs of these interpolating functions are given in Fig. 2.19.

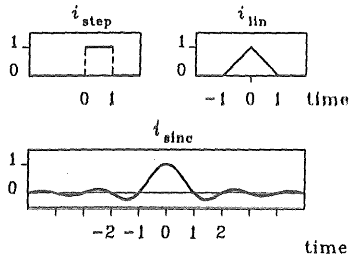


Figure 2.19 Interpolating functions. Top left: step interpolating function. Top right: linear interpolating function. Bottom: sinc interpolating function.

2.3.11. Example: Interpolation. Let x^* be the sampled signal

$$x^*(t) = \begin{cases} \cos\left(\dfrac{\pi t}{8T}\right) & \text{for } -4T \leq t < 4T, \\ 0 & \text{otherwise}, \end{cases} \qquad t \in \mathbb{Z}(T).$$

A plot of x^* is given in Fig. 2.20(a). Interpolation with the step interpolating function results in the staircase-like continuous-time signal of Fig. 2.20(b). Interpolation with the linear interpolating function leads to the signal of Fig. 2.20(c), which is obtained by connecting the sampled values of the original discrete-time signal by straight lines. Interpolation with the sinc interpolating function, finally, yields the signal of Fig. 2.20(d), which is smooth within the interval $[-4T, 4T]$ but shows a "ripple" outside it. ∎

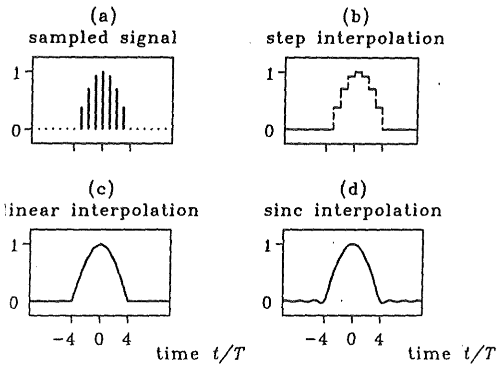


Figure 2.20 Interpolation. (a) A discrete-time signal. (b) Step interpolation. (c) Linear interpolation. (d) Sinc interpolation.

2.3.12. Remark: Are sampling and interpolation inverses of each other? Interpolation of a discrete-time signal followed by sampling on the original discrete time axis results in exact reconstruction of the original discrete-time signal. On the other hand, sampling a continuous-time signal followed by interpolation generally does not reproduce the original continuous-time signal. Evidently, sampling is an inverse operation to interpolation, but interpolation is not inverse to sampling. Figure 2.21 illustrates this. ∎

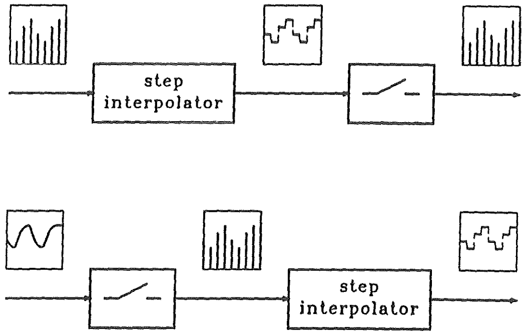


Figure 2.21 Sampling and interpolation. Top: sampling is the inverse of interpolation. Bottom: interpolation is not the inverse of sampling.

2.3.13. Remark: Analog-to-digital and digital-to-analog conversion.

(a) *A/D conversion.* Computers and other digital equipment do not only operate on a discrete time axis but are also limited to finite signal ranges. Thus, apart from sampling, conversion of real-valued continuous-time signals to input for digital equipment also involves quantization (see 2.3.3 and 2.3.4.) The combined process of sampling and quantization is called *analog-to-digital (A/D) conversion.* Figure 2.22 illustrates how a real-valued continuous-time "analog" signal is converted to a quantized discrete-time "digital" signal.

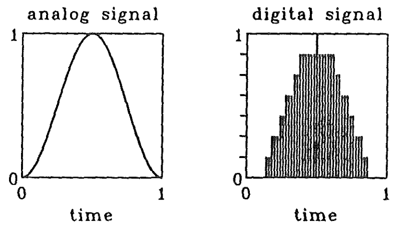


Figure 2.22 Analog-to-digital conversion. Left: analog signal. Right: digital signal.

(b) *D/A conversion.* The inverse process of converting a quantized discrete-time signal to a real-valued continuous-time signal is called *digital-to-analog (D/A) conversion.*

Devices that perform D/A conversion by step interpolation are known as *zero-order hold circuits.* They can function in "real time," meaning that given the sampled signal up to, and including, time t, the continuous-time signal up to and including that same time may be generated.

Devices that perform D/A conversion by linear interpolation are known as *first-order hold circuits*. They cannot precisely function in real time because two successive sampled signal values have to be received before it is known how the continuous-time signal goes. First-order hold circuits therefore introduce a *delay*, equal to the sampling interval T.

Sinc interpolation, finally, cannot be implemented in real time at all because *all* sampled signal values need be received before any point of the continuous-time signal (except at the sampling times) may be computed. Sinc interpolation, however, has great theoretical importance, as will be seen in Chapter 9 when we discuss the sampling theorem. ∎

Pointwise Binary Operations

We conclude this section by defining several pointwise *binary* operations among complex-valued time signals, namely *addition, subtraction, multiplication,* and *division*. All definitions are obvious extensions of the corresponding complex number operations.

2.3.14. Definition: Pointwise binary operations on complex-valued time signals. If x and y are complex-valued time signals with time axis $\mathbb{T}$, their *sum* $x + y$ and *difference* $x - y$ are again complex-valued signals with time axis $\mathbb{T}$, given by

$$(x + y)(t) = x(t) + y(t) \qquad \text{for all} \quad t \in \mathbb{T},$$
$$(x - y)(t) = x(t) - y(t) \qquad \text{for all} \quad t \in \mathbb{T}.$$

Their *product* xy is the complex-valued signal given by

$$(xy)(t) = x(t)y(t) \qquad \text{for all} \quad t \in \mathbb{T},$$

while, if $y(t) \neq 0$ for all $t \in \mathbb{T}$, the *quotient* x/y is the signal given by

$$(x/y)(t) = x(t)/y(t) \qquad \text{for all} \quad t \in \mathbb{T}.$$ ∎

All operations defined in 2.3.14 are *pointwise*, that is, the value of the resulting signal at any given time depends on the values of the two signals that are operated upon at that same time only. Note that the signals that are operated on need be defined on the *same* time axis.

2.3.15. Example: Sum and product of two time signals. Let x and y be the continuous-time signals given by

$$x(t) = t \qquad \text{for} \quad t \in \mathbb{R},$$

$$y(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0, \end{cases} \qquad t \in \mathbb{R},$$

as sketched in Fig. 2.23. Then, $x + y$ and xy are the signals

$$(x + y)(t) = \begin{cases} t & \text{for } t < 0, \\ 1 + t & \text{for } t \geq 0, \end{cases} \qquad t \in \mathbb{R},$$

$$(xy)(t) = \begin{cases} 0 & \text{for } t < 0, \\ t & \text{for } t \geq 0, \end{cases} \qquad t \in \mathbb{R},$$

also shown in Fig. 2.23.

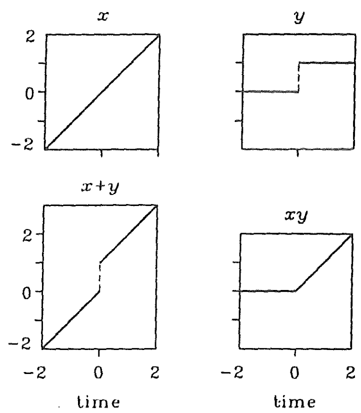


Figure 2.23 Pointwise binary operations. Top: two signals x and y. Bottom left: their sum $x + y$. Bottom right: their product xy. ∎

2.4 SIGNAL SPACES

In this section we introduce various signal sets that are important for the remainder of the book. Each of these sets has the structure of a *linear space*. A linear space, roughly, is a set in which we may *add* any two elements, resulting in another element of the set, and multiply each element by a *scalar*, again resulting in an element in the set. An axiomatic introduction to linear spaces is presented in Supplement B.

The scalars we use as multiplying factors are always *real* or *complex* numbers. In what follows, some important linear spaces are presented.

2.4.1. Example: Linear spaces.

(a) $\mathbb{R}^N$ *and* $\mathbb{C}^N$. A very well-known linear space is the set $\mathbb{R}^N$ of all N tuples $x = (x_1, x_2, \cdots, x_N)$ with $x_i \in \mathbb{R}$ for $i = 1, 2, \cdots, N$. Addition of two elements $x = (x_1, x_2, \cdots, x_N)$ and $y = (y_1, y_2, \cdots, y_N)$ is defined coordinate-wise as

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_N + y_N),$$

while multiplication of x by the real scalar α is defined as

$$\alpha x = (\alpha x_1, \alpha x_2, \cdots, \alpha x_N).$$

Thus, $\mathbb{R}^N$ is a *linear space* over the *real numbers*. Completely analogously, the set $\mathbb{C}^N$ of complex-valued N tuples is a linear space over the *complex numbers*.

(b) *The space ℓ of all time sequences*. The set ℓ of all complex-valued time sequences $x = (\cdots, x(-1), x(0), x(1), x(2), \cdots)$ is a linear space over the complex numbers if addition and multiplication by a complex number are defined pointwise as

$$(x + y)(n) = x(n) + y(n), \qquad n \in \mathbb{Z},$$
$$(\alpha x)(n) = \alpha x(n), \qquad n \in \mathbb{Z}.$$

This space is an obvious extension of $\mathbb{C}^N$.

(c) *The space $\mathcal{L}$ of all continuous-time signals*. The set of all complex-valued continuous-time signals $\mathcal{L}$ forms a linear space under pointwise addition and pointwise multiplication by a complex scalar:

$$(x + y)(t) = x(t) + y(t), \qquad t \in \mathbb{R},$$
$$(\alpha x)(t) = \alpha x(t), \qquad t \in \mathbb{R}.$$

■

Norms

It often is useful to have a measure of the *size* of a signal. The *norm* provides such a measure. The norm $\|x\|$ of an element x of a linear space X is a non-negative real number that is zero if and only if x is the zero element. The norm is *homogeneous with respect to scaling* (i.e., $\|\lambda x\| = |\lambda| \cdot \|x\|$ for every scalar λ and for every x) and satisfies the *triangle inequality* (i.e., $\|x + y\| \leq \|x\| + \|y\|$ for every x and y). A brief review of the theory of norms is given in Supplement B.

We first present a well-known family of norms on the spaces $\mathbb{R}^N$ and $\mathbb{C}^N$.

2.4.2. Example: Norms on $\mathbb{R}^N$ and $\mathbb{C}^N$. Let p be a real number such that $1 \leq p \leq \infty$. Then, the *p-norm* of an element $x = (x_1, x_2, \cdots, x_N)$ of $\mathbb{R}^N$ or $\mathbb{C}^N$ is defined as

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{N} |x_i|^p\right)^{1/p} & \text{for } 1 \leq p < \infty, \\ \max_{1 \leq i \leq N} |x_i| & \text{for } p = \infty. \end{cases}$$

The most frequently used norms are $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$. If, for instance, $x \in \mathbb{C}^2$ is given by $x = (1, j)$, then

$$\|x\|_1 = 1 + 1 = 2,$$

$$\|x\|_2 = \sqrt{1 + 1} = \sqrt{2},$$
$$\|x\|_\infty = \max(1, 1) = 1.$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the 2-norm $\|x\|_2$ is actually the *length* of the vector with coordinates x. The 2-norm is often called the *Euclidean* norm. ■

> About the life of the Greek mathematician Euclid (ca 365–ca 300 BC) not much is known for certain. His main work *Elements* has a strong logical structure and was an example for all serious mathematical treatises until the 19th century.

The p-norm may be generalized to signal spaces.

2.4.3. Example: Norms on the signal spaces ℓ and $\mathcal{L}$.

The p-norm $\|x\|_p$ of an element x of the space ℓ of time sequences is defined as

$$\|x\|_p = \begin{cases} \left(\sum_{n=-\infty}^{\infty} |x(n)|^p\right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{n \in \mathbb{Z}} |x(n)|, & p = \infty. \end{cases}$$

The p-norm $\|x\|_p$ of an element x of the space $\mathcal{L}$ of continuous-time signals is defined as

$$\|x\|_p = \begin{cases} \left(\int_{-\infty}^{\infty} |x(t)|^p \, dt\right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{t \in \mathbb{R}} |x(t)|, & p = \infty. \end{cases}$$

■

In 2.4.3, "sup" denotes the *supremum* or *least upper bound*, that is, $\sup_{n \in \mathbb{Z}} |x(n)|$ is the smallest real number α such that $|x(n)| \leq \alpha$ for all $n \in \mathbb{Z}$, if any such α exists, and ∞ if no such α exists.

The signal norms that we use in this text are the 1-, 2- and ∞-norms. The ∞-norm, given by

$$\|x\|_\infty = \sup_{n \in \mathbb{Z}} |x(n)|,$$

or

$$\|x\|_\infty = \sup_{t \in \mathbb{R}} |x(t)|,$$

is called the *amplitude* of the signal x. It is the largest magnitude the signal assumes. The square of the 2-norm, given by

$$\|x\|_2^2 = \sum_{n=-\infty}^{\infty} |x(n)|^2,$$

or

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2\, dt,$$

is referred to as the *energy* of the signal. The name derives from the fact that in electrical and mechanical systems the physical energy associated with a signal may often be expressed by a quadratic integral. Finally, the 1-norm

$$\|x\|_1 = \sum_{n=-\infty}^{\infty} |x(n)|,$$

or

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\, dt,$$

is called the *action* of the signal x in this text, because it may be viewed as a measure for the total action associated with the signal. Figure 2.24 illustrates the definitions of the amplitude, energy, and action of a signal.

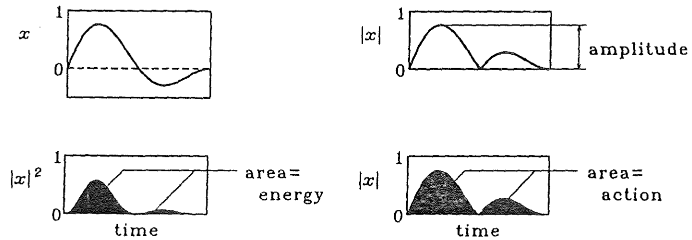


Figure 2.24 Amplitude, energy, and action of a continuous-time signal.

2.4.4. Remark: Power, rms value, and mean of a signal. Besides the amplitude, energy, and action, sometimes other quantities are used to represent certain aspects of the signal. If the energy of the signal x is not finite, its (*average*) *power*, if it exists, is defined by

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} |x(n)|^2,$$

or

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt,$$

in the discrete- and continuous-time case, respectively. The square root of the power of x is the *rms* (root mean square) value of x. The *mean* of x, finally, defined as

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} x(n),$$

or

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt,$$

respectively.

It is easily verified that the continuous-time periodic square wave of Fig. 2.25 with amplitude a has mean $\frac{1}{2}a$, power $\frac{1}{2}a^2$ and rms value $\frac{1}{2}a\sqrt{2}$.

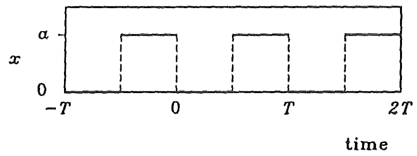


Figure 2.25 Periodic square wave. ∎

Normed Spaces

The set ℓ of all time sequences is a linear space. Not every element x of ℓ, however, has finite amplitude $\|x\|_\infty$. The subset

$$\ell_\infty := \{x \in \ell \,|\, \|x\|_\infty < \infty\}$$

of ℓ consists of all time sequences with finite amplitude. It is easy to prove that addition of finite-amplitude sequences and multiplication of a finite-amplitude sequence by a scalar result in sequences that have again finite amplitude. Hence, ℓ_∞ is a linear space. Linear spaces whose elements all have finite norms are called *normed spaces*. Thus, ℓ_∞ is a normed space, but ℓ is not. Likewise, the set $\mathscr{L}_\infty$ of all continuous-time signals with finite amplitude is a normed space.

It may be shown that also the subsets of all discrete- and continuous-time signals with *finite energy* or *finite action* are normed spaces. This result, which is elaborated on in Supplement B, may be summarized as follows.

2.4.5. Summary: Normed signal spaces.

The set

$$\ell_\infty := \{x \in \ell \,|\, \|x\|_\infty < \infty\}$$

The set

$$\mathscr{L}_\infty := \{x \in \mathscr{L} \,|\, \|x\|_\infty < \infty\}$$

of all *finite-amplitude* time sequences, the set

$$\ell_2 := \{x \in \ell \,|\, \|x\|_2 < \infty\}$$

of all *finite-energy* time sequences, and the set

$$\ell_1 := \{x \in \ell \,|\, \|x\|_1 < \infty\}$$

of all *finite-action* time sequences are normed spaces.

of all *finite-amplitude* continuous-time signals, the set

$$\mathcal{L}_2 := \{x \in \mathcal{L} \,|\, \|x\|_2 < \infty\}$$

of all *finite-energy* continuous-time signals, and the set

$$\mathcal{L}_1 := \{x \in \mathcal{L} \,|\, \|x\|_1 < \infty\}$$

of all *finite-action* continuous-time signals are normed spaces. ∎

Before continuing, we briefly review norms for sampled signals.

2.4.6. Review: Norms of sampled signals. The p-norm of a sampled signal x given on the discrete time axis $\mathbb{Z}(T)$ is defined as

$$\|x\|_p = \begin{cases} \left(T \sum_{t \in \mathbb{Z}(T)} |x(t)|^p\right)^{1/p} & \text{for } 1 \leq p < \infty, \\ \sup_{t \in \mathbb{Z}(T)} |x(t)| & \text{for } p = \infty. \end{cases}$$

In particular, the *amplitude, energy,* and *action* of x are given by

$$\|x\|_\infty = \sup_{t \in \mathbb{Z}(T)} |x(t)|,$$

$$\|x\|_2^2 = T \sum_{t \in \mathbb{Z}(T)} |x(t)|^2,$$

$$\|x\|_1 = T \sum_{t \in \mathbb{Z}(T)} |x(t)|,$$

respectively. The reason for including the factor T in the definition of the energy and action is that the resulting expressions are approximating sums for the integrals

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt \qquad \text{and} \qquad \int_{-\infty}^{\infty} |x(t)| \, dt$$

that determine the energy and action of the underlying continuous-time signal (see Fig. 2.26). The approximation improves as T decreases.

If the sampling interval T equals 1, the sampled signal x may be viewed as a time sequence; consistently with this, the p-norm of x reduces to the p-norm of the time sequence as defined in 2.4.3.

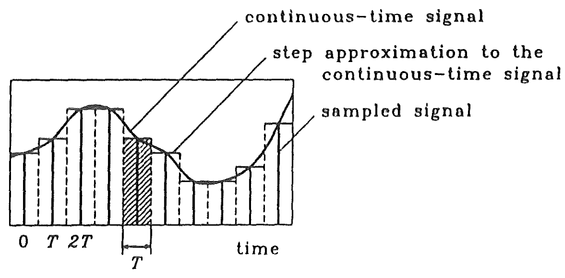


Figure 2.26 The energy and action of a sampled signal are step approximations to the energy and action of the continuous-time signal.

The sets

$$\ell_\infty(T) := \{x \in \ell(T) \,|\, \|x\|_\infty < \infty\},$$
$$\ell_2(T) := \{x \in \ell(T) \,|\, \|x\|_2 < \infty\},$$
$$\ell_1(T) := \{x \in \ell(T) \,|\, \|x\|_1 < \infty\},$$

of all finite-amplitude, finite-energy, and finite-action sampled signals on the time axis $\mathbb{Z}(T)$, respectively, are normed spaces. ∎

Inner Product

The final topic covered in this section is the notion of *inner product*. We first discuss the idea for vectors in a plane. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be the rectangular coordinates of two vectors in a plane and denote by ϕ_1 and ϕ_2 the angles that the vectors make with the horizontal axis (see Fig. 2.27). Then, the cosine of the angle $\phi_1 - \phi_2$ between the two vectors is

$$\begin{aligned} \cos(\phi_1 - \phi_2) &= \cos(\phi_1)\cos(\phi_2) + \sin(\phi_1)\sin(\phi_2) \\ &= \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \cdot \frac{y_1}{\sqrt{y_1^2 + y_2^2}} + \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \cdot \frac{y_2}{\sqrt{y_1^2 + y_2^2}} \\ &= \frac{x_1 y_1 + x_2 y_2}{(x_1^2 + x_2^2)^{1/2}(y_1^2 + y_2^2)^{1/2}}. \end{aligned}$$

The denominator of the final expression is the product of the *lengths* of the vectors, while the numerator $x_1y_1 + x_2y_2$ is known as the *scalar product* of x and y. Writing $\|x\|_2 = (x_1^2 + x_2^2)^{1/2}$ for the length of x and $\|y\|_2 = (y_1^2 + y_2^2)^{1/2}$ for that of y, we see that the scalar product of x and y is given by

$$x_1y_1 + x_2y_2 = \|x\|_2 \cdot \|y\|_2 \cos(\phi_1 - \phi_2).$$

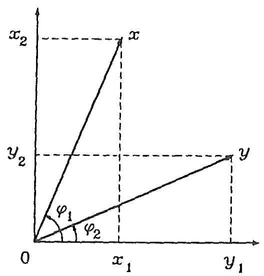


Figure 2.27 Alignment of two vectors in a plane.

For given lengths of the vectors, the scalar product is a measure for the *alignment* of the two vectors. If the scalar product is 0, $\cos(\phi_1 - \phi_2) = 0$ so that the angle between the vectors is 90°, which means that the vectors are perpendicular. The scalar product is maximal when $\cos(\phi_1 - \phi_2) = 1$, which means that the angle between the vectors is 0.

The scalar product of two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in *three*-dimensional space is $x_1y_1 + x_2y_2 + x_3y_3$ and has a geometric interpretation similar to that of the scalar product of two-dimensional vectors.

The *inner product* $\langle x, y\rangle$ of two elements x and y of a linear space is a generalization of the scalar product. An axiomatic definition is given in Supplement B. An *inner product space* is a linear space such that any two elements have a finite inner product.

We first consider inner products on $\mathbb{R}^N$ and $\mathbb{C}^N$.

2.4.7. Examples: Inner product on $\mathbb{R}^N$ and $\mathbb{C}^N$.

(a) *Inner product on* $\mathbb{R}^N$. The inner product

$$\langle x, y\rangle = \sum_{i=1}^{N} x_i y_i$$

of two elements $x = (x_1, x_2, \cdots, x_N)$ and $y = (y_1, y_2, \cdots, y_N)$ of $\mathbb{R}^N$ is a direct generalization of the scalar product.

(b) *Inner product on* $\mathbb{C}^N$. The elements $x = (x_1, x_2, \cdots, x_N)$ and $y = (y_1, y_2, \cdots, y_N)$ of $\mathbb{C}^N$ in general have complex coordinates. Their inner product is defined as

$$\langle x, y\rangle = \sum_{i=1}^{N} x_i \bar{y}_i,$$

with the overbar denoting the complex conjugate. The introduction of the complex conjugate guarantees the inner product $\langle x, x\rangle$ of any element x of $\mathbb{C}^N$ with itself to be real and positive. In fact, $\langle x, x\rangle = \|x\|_2^2$. ∎

The notion of inner product may easily be generalized to signal spaces. The i product of two signals is defined analogously to the inner product on $\mathbb{C}^N$.

2.4.8. Example: Inner products on signal spaces.

The inner product of two time sequences x and y belonging to the signal space ℓ is defined as

$$\langle x, y\rangle = \sum_{n\in\mathbb{Z}} x(n)\overline{y(n)}.$$

The inner product of x with itself is

$$\langle x, x\rangle = \sum_{n\in\mathbb{Z}} |x(n)|^2 = \|x\|_2^2.$$

Since two arbitrary elements x and y of ℓ may well have an infinite inner product, such as $x = y = (\cdots, 1, 1, 1, \cdots)$, ℓ is not an inner product space. It follows from the Cauchy-Schwarz inequality below that two *finite-energy* time sequences always have a finite inner product, so that the space ℓ_2 of all finite-energy sequences is an inner product space.

The inner product of two continuous-t signals x and y belonging to the si space $\mathscr{L}$ is defined as

$$\langle x, y\rangle = \int_{-\infty}^{\infty} x(t)\overline{y(t)}\, dt.$$

The inner product of x with itself is

$$\langle x, x\rangle = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \|x\|_2^2.$$

Since two arbitrary elements x and y of . may well have an infinite inner product, . is not an inner product space. It follow from the Cauchy-Schwarz inequality belo that two *finite-energy* continuous-time sig nals always have a finite inner product, s that the space $\mathscr{L}_2$ of all finite-energy con tinuous-time signals is an inner produc space. ∎

Earlier in this section we saw that if x and y are two elements of $\mathbb{R}^2$ or $\mathbb{R}^3$, then the cosine of the angle between x and y is given by

$$\cos(\phi_1 - \phi_2) = \frac{\langle x, y\rangle}{\|x\|_2 \cdot \|y\|_2}.$$

The right-hand side of this expression may vary from 1 to -1. If it is 1, then the angle between x and y is 0 and x and y are completely aligned. If the right-hand side is 0, then x and y are perpendicular. If the right-hand side is -1, then the vectors are again aligned but in opposed directions.

The *Cauchy-Schwarz inequality* allows a generalization of this argument to any inner product space.

2.4.9. Summary: The Cauchy-Schwarz inequality. For any x and y belonging to an inner product space

$$|\langle x, y\rangle|^2 \le \langle x, x\rangle \langle y, y\rangle.$$

Equality holds if and only if there exists a scalar α such that $x = \alpha y$ or $y = \alpha x$. ∎

The French mathematician Augustin Louis Cauchy (1789–1857) was trained as a civil engineer and contributed to many fields of mathematics. Hermann Amandus Schwarz (1843–1921) was a German mathematician.

The proof of the Cauchy-Schwarz inequality may be found in Supplement B. Recalling that in the various inner product spaces we considered the inner product $\langle x, x\rangle$ of x with itself equals $\|x\|_2^2$, for real spaces we may rewrite the Cauchy-Schwarz inequality in the form

$$-1 \leq \frac{\langle x, y\rangle}{\|x\|_2 \cdot \|y\|_2} \leq 1.$$

Interpreting

$$\frac{\langle x, y\rangle}{\|x\|_2 \cdot \|y\|_2}$$

as the "cosine of the angle between x and y" we see that x and y are maximally aligned (in the same or opposed directions) if their inner product $\langle x, y\rangle$ equals $\pm\|x\|_2 \cdot \|y\|_2$, while they are least aligned when their inner product is 0.

2.4.10. Review: Inner product of sampled signals. The inner product of two sampled signals x and y on the discrete time axis $\mathbb{Z}(T)$ is defined by

$$\langle x, y\rangle = T \sum_{t \in \mathbb{Z}(T)} x(t)\overline{y(t)}.$$

For $T = 1$ the definition reduces to the inner product of sequences. The set $\ell_2(T)$ of all finite-energy sampled signals on the time axis $\mathbb{Z}(T)$ is an inner product space. ■

2.4.11. Application: Signal recognition. In this example we show how the idea of measuring the alignment of two signals by their inner product may be used in signal recognition. Suppose that we receive a continuous-time signal z along some transmission channel that is either a "long pulse" or a "short pulse." A "long pulse" is the signal l defined by

$$l(t) = \begin{cases} 1/2 & \text{for } -2 \leq t < 2, \\ 0 & \text{otherwise,} \end{cases}$$

while a "short pulse" is the signal s given by

$$s(t) = \begin{cases} 1 & \text{for } -1/2 \leq t < 1/2, \\ 0 & \text{otherwise} \end{cases}$$

(see Fig. 2.28). The pulses have been scaled such that their energies and, hence, also their 2-norms, are 1.

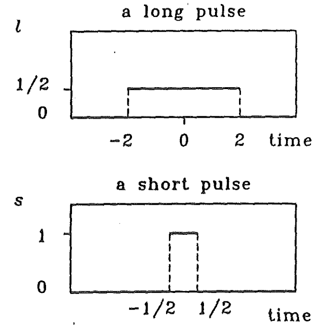


Figure 2.28 Top: a long pulse. Bottom: a short pulse.

One way of detecting whether the received signal z is a long or short pulse is to take the inner product of z with both l and s and to see with which z is best aligned. Suppose that z is a long pulse. Then,

$$\langle z, l\rangle = \langle l, l\rangle = 1,$$
$$\langle z, s\rangle = \langle l, s\rangle = 1/2,$$

so that z is best aligned with the long pulse l, which is not surprising. On the other hand, if z is a short pulse,

$$\langle z, l\rangle = \langle s, l\rangle = 1/2,$$
$$\langle z, s\rangle = \langle s, s\rangle = 1,$$

so that z is best aligned with the short pulse s.

We may thus use the following signal recognition scheme: If z is better aligned with the long pulse l than with the short pulse s, that is, if

$$\langle z, l\rangle > \langle z, s\rangle,$$

then the signal z is recognized as a long pulse, while, if it is better aligned with the short pulse s than with the long pulse l, that is, if

$$\langle z, l\rangle < \langle z, s\rangle,$$

then it is recognized as a short pulse.

This method of detecting which signal was received may also be used if the signal were distorted during transmission. Because taking the inner product is an averaging operation, the method often yields the correct answer. One of the results of advanced communication theory is that under wide assumptions concerning the nature of the distortion taking inner products is the best way of identifying the signal. ∎

2.5 GENERALIZED SIGNALS

All continuous-time signals we encountered so far were real- or complex-valued functions of a real variable. From now on we call such signals *regular* signals, because, in this section, we enlarge our repertoire with what are called *singular* signals. These are *not* functions in the ordinary sense. Together, the regular and singular signals form the *generalized* signals.

The Need for the Delta Function

To explain why singular signals are needed, consider charging a capacitor. In the circuit of Fig. 2.29 the switch initially is in the lower position and the capacitor uncharged. At time 0 the switch is set to the upper position. Because by assumption the resistance in the circuit is very small, a "burst" of current flows from the battery to the capacitor, which is almost instantaneously charged to the voltage V of the battery. Figure 2.30 shows the current i to the capacitor and its voltage. Often it is not important to know the precise shape of the current burst: all that is relevant is that the capacitor has been charged with charge $Q = CV$. Whatever the shape of the current burst is,

$$\int_{-\infty}^{\infty} i(t)\, dt = CV \tag{1}$$

holds.

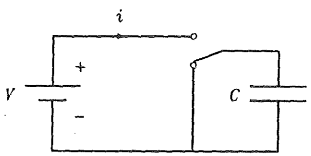


Figure 2.29 Charging a capacitor.

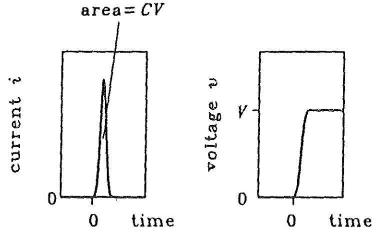


Figure 2.30 Left: the current to the capacitor. Right: the voltage across the capacitor.

If one insists on asking what the values of i are as a function of time under the assumption of *zero* resistance, it is tempting to answer

$$i(t) = \begin{cases} \infty & \text{for } t = 0, \\ 0 & \text{otherwise}, \end{cases} \qquad t \in \mathbb{R}.$$

The difficulty is that

(i) this is *not* a function in the ordinary sense because $\infty \notin \mathbb{C}$, and
(ii) since the current i equals 0 everywhere except at one point of time, its integra must be 0, contradicting (1).

Evidently, we need revise our notion of what a signal is if we wish to include the current burst in our repertoire. To this end, we enlarge the set of regular signals with a *singular* signal δ, called *delta function*, with the properties

$$\delta(t) = 0 \qquad \text{for} \qquad t \neq 0,$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

This signal is an "infinitely short, infinitely large pulse at time zero, with area one." Figure 2.31 gives a symbolic graphical representation of the δ-function. With this function we may describe the current that charges the capacitor as

$$i(t) = CV\,\delta(t), \qquad t \in \mathbb{R}.$$

The delta function is useful for describing other practical situations as well.

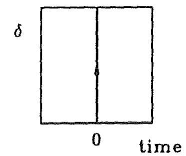


Figure 2.31 Symbolic representation of a δ-function.

2.5.1. Example: Mechanical impact. An example of a mechanical phenomenon for which the δ-function is a useful model is the *impact* of a hammer that hits a nail. The force F exerted on the nail is very large and has very short duration, but the impact

$$P = \int_{-\infty}^{\infty} F(t)\, dt$$

is finite. The penetration of the nail is proportional to the impact. The "time behavior" of the force may be modeled as

$$F(t) = P\delta(t), \qquad t \in \mathbb{R}.$$ ∎

2.5.2. Example: Point masses and point charges. If we denote by ρ the mass or charge density along some spatial axis, a convenient model to describe a *point* charge or mass is

$$\rho(x) = A\delta(x), \qquad x \in \mathbb{R},$$

with A denoting the point charge Q or the point mass M, respectively. ∎

2.5.3. Exercise: Approximation to the δ-function. Consider the circuit of Fig. 2.32, which is that of Fig. 2.29 with a small resistance R included that represents the resistance of the wires.

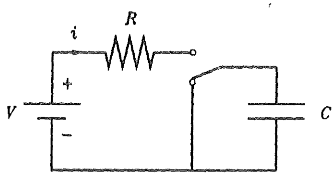


Figure 2.32 Charging a capacitor with a small resistance.

(a) Prove that if the switch is thrown from the lower to the upper position at time 0, the current is given by

$$i_R(t) = \begin{cases} 0 & \text{for } t < 0, \\ \dfrac{V}{R}e^{-t/RC} & \text{for } t \geq 0, \end{cases} \qquad t \in \mathbb{R},$$

as shown in Fig. 2.33, and that

$$\int_{-\infty}^{\infty} i_R(t)\,dt = CV.$$

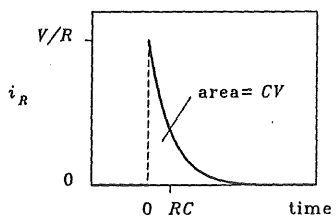


Figure 2.33 The current i_R.

(b) Show that

(i)

$$\lim_{R\to 0} i_R(t) = \begin{cases} 0 & \text{for } t \neq 0, \\ \infty & \text{for } t = 0, \end{cases}$$

(ii)

$$\lim_{R\to 0} \int_{-\infty}^{\infty} i_R(t)\,dt = CV,$$

(iii)

$$\int_{-\infty}^{\infty} \left(\lim_{R\to 0} i_R(t)\right) dt = 0.$$

Evidently, for small R the function i_R approximates the δ-function (within a factor CV), but $\lim_{R\to 0} i_R$ is *not* the δ-function, because of (iii). ∎

Principles of the Theory of Singular Functions

A delta function is an example of a singular signal. Singular signals are signals that cannot be treated "pointwise," such as ordinary signals. As we have seen, a delta function has zero width and infinite height. Only its area, which is 1, is finite. The correct way to introduce delta functions, therefore, is "under the integral sign." We thus *define* the delta function δ as the entity such that

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt = \phi(0)$$

for every regular function ϕ that is continuous at 0. We shall see that even though there is *no* such regular function δ, this definition allows us to introduce various operations on the delta function and establish its properties.

The delta function is not the only singular function; in particular, we shall soon meet the *derivatives* of the delta function.

The mathematical theory of singular functions is called *distribution theory*. A brief outline of this theory is given in Supplement C. In what follows we present an abbreviated exposition that introduces all the singular signals we need, and describes how to work with them.

> Delta functions originally arose in theoretical physics, but their mathematical justification remained debatable for some time. Their theory was put on a firm mathematical basis in the period 1945–1950 by the French mathematician Laurent Schwartz (b. 1915), who developed distribution theory.

2.5.4. Definition: Delta function. The delta function δ is the singular function such that

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)\, dt = \phi(0)$$

for every regular function ϕ that is continuous at 0.

Thus, for instance,

$$\int_{-\infty}^{\infty} e^{-t^2}\delta(t)\, dt = 1,$$

because the regular function ϕ given by

$$\phi(t) = e^{-t^2}, \qquad t \in \mathbb{R},$$

is continuous at 0 and equals 1 at this point.

In the sequel we encounter various other singular signals, such as shifted delta functions and derivatives of delta functions. Singular signals are defined "under the integral sign." This means that the *equality* of two generalized signals ε_1 and ε_2 need be established by verifying that

$$\int_{-\infty}^{\infty} \varepsilon_1(t)\phi(t)\, dt = \int_{-\infty}^{\infty} \varepsilon_2(t)\phi(t)dt$$

for all regular functions ϕ that are continuous and differentiable wherever and as often as needed.

Linear Combinations of Delta Functions

Equipped with the basic definition of a δ-function and the rule that singular functions are equal if they produce the same answer "under the integral sign," we now set out to define various operations on the delta function, such as linear combination, time scaling and time shifting, and differentiation. These lead to new singular signals. The operations are so defined that they are entirely consistent with the rules for regular functions.

The first of the operations we consider are *addition* of singular signals and *multiplication by a scalar*. For *regular* signals f_1 and f_2 and scalars α_1 and α_2 we have

$$\int_{-\infty}^{\infty} (\alpha_1 f_1 + \alpha_2 f_2)(t)\phi(t)\, dt = \alpha_1 \int_{-\infty}^{\infty} f_1(t)\phi(t)\, dt + \alpha_2 \int_{-\infty}^{\infty} f_2(t)\phi(t)\, dt$$

for every function ϕ such that the integrals exist. Hence, for singular signals ε_1 and ε_2 and scalars α_1 and α_2 we *define*

$$\int_{-\infty}^{\infty} (\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2)(t)\phi(t)\, dt = \alpha_1 \int_{-\infty}^{\infty} \varepsilon_1(t)\phi(t)\, dt + \alpha_2 \int_{-\infty}^{\infty} \varepsilon_2(t)\phi(t)\, dt$$

for every ϕ that is continuous and differentiable as often as needed.

In particular, the product $\alpha\delta$ of the scalar α and the δ-function is definec

$$\int_{-\infty}^{\infty} (\alpha\delta)(t)\phi(t)\, dt = \alpha \int_{-\infty}^{\infty} \delta(t)\phi(t)\, dt = \alpha\phi(0)$$

for every function ϕ that is continuous at 0.

2.5.5. Exercise: Sum of δ-functions. Prove that

$$\alpha\delta + \beta\delta = (\alpha + \beta)\delta$$

for any complex scalars α and β.

Time Scaling and Time Translation of Delta Functions

We next study how the δ-function may be scaled in time. Let α be a nonzero re number. If f is a regular function, then it follows by the substitution $\alpha t = \tau$ that

$$\int_{-\infty}^{\infty} f(\alpha t)\phi(t)\, dt = \frac{1}{|\alpha|}\int_{-\infty}^{\infty} f(\tau)\phi\left(\frac{\tau}{\alpha}\right) d\tau$$

for every ϕ such that the integral exists. For singular signals we take this to be th *definition* of time scaling. It follows for the delta function

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(\alpha t)\phi(t)\, dt &= \frac{1}{|\alpha|}\int_{-\infty}^{\infty} \delta(\tau)\phi\left(\frac{\tau}{\alpha}\right) d\tau \\ &= \frac{1}{|\alpha|}\phi(0),\end{aligned}$$

where second step follows by the basic property of the δ-function for any ϕ that is continuous at 0. We recognize that

$$\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t), \qquad t \in \mathbb{R},$$

because application of the left-hand side to any function ϕ that is continuous at 0 gives the same result as applying the right-hand side.

We now consider how the δ-function may be translated in time. Let θ be a real number. Then, if δ were a regular function, it would follow by the substitution

$t - \theta = \tau$ that for every function ϕ that is continuous at θ

$$\int_{-\infty}^{\infty} \delta(t-\theta)\phi(t)\,dt = \int_{-\infty}^{\infty} \delta(\tau)\phi(\tau+\theta)\,d\tau$$
$$= \phi(\theta),$$

where the final step again follows from the basic property of the δ-function. Thus, we define the *translated δ-function* as the singular function such that

$$\int_{-\infty}^{\infty} \delta(t-\theta)\phi(t)\,dt = \phi(\theta)$$

for every regular function ϕ that is continuous at θ. The translated δ-function $\delta(t-\theta)$, $t \in \mathbb{R}$, may graphically be represented as in Fig. 2.34.

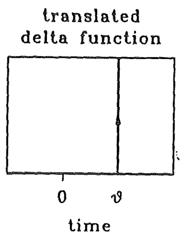


Figure 2.34 A translated δ-function.

Multiplication by a Function

We next consider the effect of multiplying the δ-function by a time function $f(t)$, $t \in \mathbb{R}$. If δ were a regular function, we would have for any ϕ that is continuous at 0

$$\int_{-\infty}^{\infty} [f(t)\delta(t)]\phi(t)\,dt = \int_{-\infty}^{\infty} \delta(t)[f(t)\phi(t)]\,dt$$
$$= f(0)\phi(0),$$

where the final step again follows by the basic property of the δ function, *provided* f is continuous at 0. Thus, we *define* the product $f(t)\delta(t)$, $t \in \mathbb{R}$, of a function f that is continuous at 0 and the δ-function as the singular function such that

$$\int_{-\infty}^{\infty} [f(t)\delta(t)]\phi(t)\,dt = f(0)\phi(0)$$

for every function ϕ that is continuous at 0. Inspection shows that

$$f(t)\delta(t) = f(0)\delta(t), \qquad t \in \mathbb{R}.$$

The assumption that f be continuous at 0 is essential. *A fortiori*, f cannot be singular at 0. In particular, the δ-function cannot be multiplied by itself.

Differentiation

We now consider how δ-functions may be differentiated. If f is a regular function, then it follows by partial integration that

$$\int_{-\infty}^{\infty} \frac{df(t)}{dt}\phi(t)\,dt = f(t)\phi(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\frac{d\phi(t)}{dt}\,dt$$
$$= -\int_{-\infty}^{\infty} f(t)\frac{d\phi(t)}{dt}\,dt$$

for any ϕ such that the integral exists and such that $f(-\infty)\phi(-\infty) = f(\infty)\phi(\infty) = 0$. If f were the δ-function, the right-hand side would reduce to

$$\int_{-\infty}^{\infty} \delta(t)\frac{d\phi(t)}{dt}\,dt = -\phi^{(1)}(0),$$

where $\phi^{(1)}$ stands for the derivative of ϕ. Hence, we *define* the derivative $\delta^{(1)}$ of the delta function by

$$\int_{-\infty}^{\infty} \delta^{(1)}(t)\phi(t)\,dt = -\phi^{(1)}(0)$$

for every function ϕ that is continuously differentiable at 0.

In the same way we may determine the nth derivative $\delta^{(n)}$ of the δ-function. By repeated partial integration it easily follows that $\delta^{(n)}$ need be defined by the requirement that

$$\int_{-\infty}^{\infty} \delta^{(n)}(t)\phi(t)\,dt = (-1)^n\phi^{(n)}(0)$$

for every regular function ϕ that is n times continuously differentiable at 0.

2.5.6. Exercise. Product of the nth derivative of a δ-function and a function. Prove that if $f(t)$, $t \in \mathbb{R}$, is a function whose nth derivative is continuous at 0,

$$f(t)\delta^{(n)}(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f^{(k)}(0)\delta^{(n-k)}(t), \qquad t \in \mathbb{R}.$$

In particular,

$$f(t)\delta^{(1)}(t) = f(0)\delta^{(1)}(t) - f^{(1)}(0)\delta(t), \qquad t \in \mathbb{R}.$$

Delta Function as the Derivative of a Step

Another way of looking at the δ-function is to consider it as the derivative of a unit step. In the ordinary sense, the step function $\mathbb{1}$ does not have a derivative, but, if it had, then by partial integration we would have for any differentiable function ϕ

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\mathbb{1}}{dt}(t)\phi(t)\,dt &= \mathbb{1}(t)\phi(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \mathbb{1}(t)\phi^{(1)}(t)\,dt \\ &= \phi(\infty) - \int_{0}^{\infty} \phi^{(1)}(t)\,dt = \phi(\infty) - \phi(t)\Big|_0^{\infty} = \phi(0) \\ &= \int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt.\end{aligned}$$

This proves that in the context of generalized functions the derivative of the unit step is the δ-function. Figure 2.35 illustrates this.

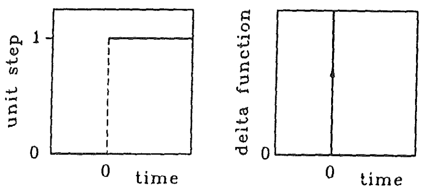


Figure 2.35 The derivative of the unit step is the δ-function. Left: the unit step. Right: its derivative.

The great advantage of the theory of singular signals is that any signal, including signals with discontinuities and singularities, may be differentiated arbitrarily often.

2.5.7. Example: Acceleration of a mass. Consider a unit mass that moves along a straight line. Suppose that we wish its position s to depend on time as

$$s(t) = \begin{cases} 0 & \text{for } t < 0, \\ t & \text{for } t \geq 0, \end{cases}$$

that is, the mass is to be at rest until time 0, and to move at a constant speed after time 0. The speed v and acceleration a of the mass follow by successive differentiation as

$$\begin{aligned} v(t) &= \frac{ds(t)}{dt} = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0, \end{cases} \\ &= \mathbb{1}(t), \qquad t \in \mathbb{R}, \end{aligned}$$

$$a(t) = \frac{dv(t)}{dt} = \delta(t), \qquad t \in \mathbb{R}.$$

By Newton's law, this means that the force F applied to the mass need have the pulse-like behavior

$$F(t) = \delta(t), \qquad t \in \mathbb{R}.$$

Figure 2.36 shows the position, velocity, and acceleration of the mass.

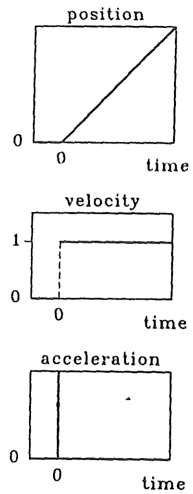


Figure 2.36 A moving mass. Top: its position. Middle: its velocity. Bottom: its acceleration.

Approximations to Delta Functions

The δ-function and its derivatives do not exist as regular functions. They may be *approximated* by sequences of regular functions, however. Define, for instance, the sequence of functions d_n given by

$$d_n(t) = \begin{cases} n & \text{for } -1/2n \leq t < 1/2n, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{R},$$

as shown in Fig. 2.37 (left). This sequence approximates the δ-function in the sense that for every function ϕ that is continuous at 0

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} d_n(t)\phi(t)\,dt = \phi(0).$$

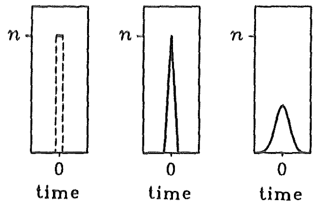


Figure 2.37 Approximations to the δ-function. Left: rectangular approximation. Middle: triangular approximation. Right: bell-shaped approximation.

Note, however, that

$$\lim_{n\to\infty} d_n$$

does *not* exist, which is consistent with the fact that the δ-function is not a regular function.

Other well-known approximations to the δ-function are

$$d_n(t) = n\,\text{trian}(nt), \qquad t \in \mathbb{R},$$

and

$$d_n(t) = n\,\text{bell}(nt), \qquad t \in \mathbb{R}.$$

The "bell" function is defined as

$$\text{bell}(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \qquad t \in \mathbb{R}.$$

Plots of these approximations are also given in Fig. 2.37.

Approximations to the derivatives $\delta^{(k)}$ of the δ-function may be obtained by differentiating sufficiently smooth approximations to the δ-function, for instance, the bell-shaped approximation. Figure 2.38 shows approximations to the first derivative $\delta^{(1)}$ and the second derivative $\delta^{(2)}$, obtained by differentiating the bell approximation. Because of the shape of the approximation, $\delta^{(1)}$ is sometimes called a "doublet" function and $\delta^{(2)}$ a "triplet" function. In physics, the conjunction of an infinite positive and negative charge is known as an electrical *dipole*. Its mathematical description is a doublet.

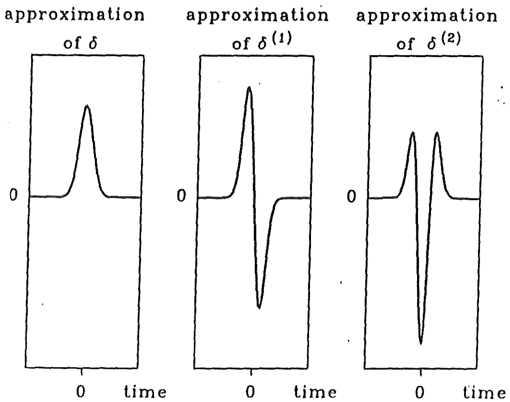


Figure 2.38 Approximations to δ-functions. Left: bell approximation to the δ-function. Middle: approximation to the first derivative $\delta^{(1)}$. Right: approximation to the second derivative $\delta^{(2)}$.

Review

Table 2.1 Summarizes the main properties of the δ-function.

TABLE 2.1 PROPERTIES OF THE δ-FUNCTION

	Property	Conditions
(1)	$\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt = \phi(0)$	ϕ continuous at 0
(2)	$\delta(\alpha t) = \frac{1}{\lvert\alpha\rvert}\delta(t)$	α real and nonzero
(3)	$\int_{-\infty}^{\infty} \delta(t-\theta)\phi(t)\,dt = \phi(\theta)$	θ real, ϕ continuous at θ
(4)	$f(t)\delta(t) = f(0)\delta(t)$	f continuous at 0
(5)	$\frac{d}{dt}\mathbb{1}(t) = \delta(t)$	
(6)	$\frac{d^n}{dt^n}\delta(t) = \delta^{(n)}(t)$	$n \in \mathbb{N}$
(7)	$\int_{-\infty}^{\infty} \delta^{(n)}(t)\phi(t)\,dt = (-1)^n\phi^{(n)}(0)$	$n \in \mathbb{Z}_+$, ϕ at least n times continuously differentiable at 0
(8)	$f(t)\delta^{(n)}(t) = \sum_{k=0}^{n}(-1)^k\binom{n}{k} f^{(k)}(0)\delta^{(n-k)}(t)$	$n \in \mathbb{Z}_+$, f at least n times continuously differentiable at 0

2.6 PROBLEMS

The first few problems deal with complex numbers, sets, maps, and power sets as reviewed in Supplement A.

2.6.1. Complex numbers. Given the following pairs of complex numbers x and y, compute their sum $x + y$, product xy, quotient x/y as well as the exponential e^x. Represent the results both in Cartesian and in polar form.

(a) $x = 1 + j$, $y = 1 - j$.
(b) $x = e^{j\pi/2}$, $y = e^{-j\pi/2}$.
(c) $x = e^{j\pi/4}$, $y = 1 + j$.

2.6.2. Complex conjugate. If x and y are complex numbers, prove the following:

(a) $\overline{x \pm y} = \overline{x} \pm \overline{y}$.
(b) $\overline{xy} = \overline{x}\,\overline{y}$ and $\overline{(x/y)} = \overline{x}/\overline{y}$.

2.6.3. Product set. Of which sets A and B is the set C = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)} the product set?

2.6.4. Power sets.

(a) Give examples of elements of the power sets $\mathbb{B}^{\mathbb{B}}$ and $\mathbb{Z}^{\mathbb{B}}$, where $\mathbb{B}$ is the set $\mathbb{B} = \{0, 1\}$. Show that $\mathbb{B}^{\mathbb{B}}$ has four elements and that $\mathbb{Z}^{\mathbb{B}}$ is countably infinite.
(b) Suppose that X is a finite set containing N elements and Y a finite set with M elements. Prove that the power set X^Y has N^M elements.
(c) Show that the power sets $\ell = \mathbb{C}^{\mathbb{Z}}$ and $\mathcal{L} = \mathbb{C}^{\mathbb{R}}$ are uncountable.

2.6.5. Maps. Verify whether the following maps $\phi: X \to Y$ are surjective, injective, or bijective. If the map ϕ is bijective, determine its inverse.

(a) $X = \mathbb{R}$, $Y = \mathbb{R}$, $\phi(x) = \sin(x)$.
(b) $X = \mathbb{R}$, $Y = [-\pi/2, \pi/2]$, $\phi(x) = \operatorname{atan}(x)$.
(c) $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$, $\phi(x_1, x_2) = (2x_1, x_1 + x_2)$.
(d) $X = \mathbb{R}_+$, $Y = \mathbb{R}_+$, $\phi(x) = \sqrt{x}$.

The following problems relate to time signals as introduced in Section 2.2.

2.6.6. Various time signals. Plot the following continuous-time signals x.

(a) $x(t) = \mathbb{1}(t)\mathbb{1}(\theta - t)$, $t \in \mathbb{R}$, for some fixed $\theta \in \mathbb{R}$.
(b) $x(t) = e^{-t}\mathbb{1}(t)$, $t \in \mathbb{R}$.
(c) $x(t) = \sin(\operatorname{ramp}(t))$, $t \in \mathbb{R}$.
(d) $x(t) = \operatorname{ramp}(\sin(t))$, $t \in \mathbb{R}$.
(e) $x(t) = \operatorname{rect}(t/2 - 1)$, $t \in \mathbb{R}$.

Also, plot the following discrete-time signals.

(f) $x(n) = a^n\mathbb{1}(n)$, $n \in \mathbb{Z}$, with a a real number such that $0 < a < 1$.
(g) $x(n) = a^n\mathbb{1}(-n)$, $n \in \mathbb{Z}$, with a a real number such that $a > 1$.
(h) $x(n) = \sum_{k=0}^{\infty} \mathbb{1}(n - kN)$, $n \in \mathbb{Z}$, with N a fixed natural number.

2.6.7. Periodic time signals. Let z_1 and z_2 be the discrete-time signals defined by

$$z_1(n) = (-1)^n, \qquad n \in \mathbb{Z}, \qquad z_2(n) = j^n, \qquad n \in \mathbb{Z}.$$

(a) Plot the time signals.
(b) Show that z_1 and z_2 are periodic, and determine their periods.

2.6.8. Aliasing. Consider the complex harmonic signals η_{f_i}, $i = 1, 2, \cdots, 5$, defined on the discrete time axis $\mathbb{T} = \mathbb{Z}$, where $f_1 = 0$, $f_2 = 1/2$, $f_3 = 1$, $f_4 = 5/4$, and $f_5 = 3/2$. How many distinct harmonic signals do we really have?

2.6.9. Average repetition rate of discrete-time harmonic signals. Let c_f be the discrete-time real harmonic signal with frequency $f \in \mathbb{R}$ defined by

$$c_f(t) = \cos(2\pi f t), \qquad t \in \mathbb{Z}(T).$$

(a) Suppose that $-1/2T \leq f < 1/2T$. Define for the natural number N the quantity M_N as the number of *maxima* of the signal c_f on the interval $[-NT, NT)$. Prove that

$$\lim_{N\to\infty} \frac{M_N}{2NT} = |f|,$$

that is, the *average* number of maxima of the harmonic signal per unit of tim equals $|f|$.
(b) Suppose that f does not necessarily lie in the interval $[-1/2T, 1/2T)$. Show th the average number of maxima per unit of time now is $|f'|$, where $f' = f - k/T$ with $k \in \mathbb{Z}$ such that f' lies in the interval $[-1/2T, 1/2T)$.

This result shows that the average repetition rate of a discrete-time harmonic signa with frequency f is $|f'|$.

The following problems concern elementary operations on time signals, as presented in Section 2.3.

2.6.10. Signal range transformations. Consider the continuous-time real harmonic signal x given by

$$x(t) = \sin(2\pi f t), \qquad t \in \mathbb{R}.$$

Plot this signal after application of the following signal range transformations ρ: $\mathbb{R} \to \mathbb{R}$.

(a) *Hard limiting*: $\rho(x) = \operatorname{sign}(x)$, $x \in \mathbb{R}$, with the sign function as defined in 3.2.6(b).
(b) *Soft limiting*: $\rho(x) = \operatorname{sat}(x)$, $x \in \mathbb{R}$, with the sat function as defined in 3.2.6(b).
(c) *Half-wave rectification*: $\rho(x) = \operatorname{ramp}(x)$, $x \in \mathbb{R}$.
(d) *Squaring*: $\rho(x) = x^2$, $x \in \mathbb{R}$.

2.6.11. Time transformations. Let x be the continuous-time signal given by

$$x(t) = \sin(2\pi f t)\mathbb{1}(t), \qquad t \in \mathbb{R}.$$

(a) Plot the signal.
(b) Plot the signal after time compression by a factor 2.
(c) Plot the signal after time expansion by a factor 2.
(d) Plot the signal after time reversal.
(e) Plot the signal after translating it in time by 1 time unit.

2.6.12. Sampling and interpolation.

(a) Give an example of a continuous-time signal where sampling followed by step interpolation exactly reconstructs the original continuous-time signal.
(b) Give an example of a continuous-time signal—not that of (a)—where sampling followed by linear interpolation exactly reconstructs the original continuous-time signal.

The next few problems involve various aspects of signal spaces as discussed in Section 2.4 and Supplement B.

2.6.13. Amplitude, energy, and action. Compute the amplitude, energy, and action of the following signals.

(a) $z(t) = \dfrac{1}{1+t}\mathbb{1}(t)$, $t \in \mathbb{R}$.
(b) $z(t) = \cos(2\pi ft)$, $t \in \mathbb{R}$, for some fixed real frequency f.
(c) $z(n) = a^n\mathbb{1}(-n)$, $n \in \mathbb{Z}$, for some fixed complex number a. Carefully distinguish whether $|a| < 1$, $|a| = 1$, or $|a| > 1$.
(d) $z(n) = a^{|n|}$, $n \in \mathbb{Z}$, for some fixed complex number a. Carefully distinguish whether $|a| < 1$, $|a| = 1$, or $|a| > 1$.

2.6.14. Effect of time translation, expansion, and compression on amplitude, energy, and action.

(a) Prove that time translation does not change the amplitude, energy, and action of any time signal in $\mathcal{L}$.
(b) Show that time expansion or compression does not affect the amplitude of any signal in $\mathcal{L}$ but multiplies its energy and action by a constant.

2.6.15. Signal spaces. Determine to which of the signal sets ℓ_∞, ℓ_2, and ℓ_1 the following discrete-time signals z belong.

(a) $z(n) = \mathbb{1}(n)\mathbb{1}(N - n)$, $n \in \mathbb{Z}$, for some fixed integer N.
(b) $z(n) = a^{|n|}$, $n \in \mathbb{Z}$, with a a fixed complex number. Distinguish the cases $|a| < 1$, $|a| = 1$, and $|a| > 1$.

2.6.16. Inner product. Let $x = (1, 1, 1)$ and $y \in \mathbb{R}^3$ represent the rectangular coordinates of two vectors in three-dimensional space. Use the inner product to compute the angle between x and y in the following cases:

(a) $y = (1, 0, 0)$.
(b) $y = (1, 1, 0)$.
(c) $y = (1, 2, 3)$.

2.6.17. Signal recognition. We follow up Example 2.4.11, and suppose that the signal z is a rectangular pulse as in Fig. 2.39, which is nonzero on the interval $[-a, a]$ and has height b.

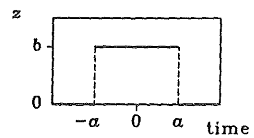


Figure 2.39 A rectangular pulse.

(a) Amplitude scale the pulse such that its energy is 1 (i.e., choose b such that the energy of the signal is 1).
(b) Compute the inner products $<z, s>$ and $<z, l>$ of the signal with the short pulse s and the long pulse l (carefully distinguish whether a lies between 0 and 1/2, between 1/2 and 2, or is greater than 2).
(c) Determine for which values of a the pulse is best aligned with the short pulse and for which values it is best aligned with the long pulse.

(d) Suppose that z is the triangular pulse $z(t) = c\ \mathrm{trian}(t/2)$, $t \in \mathbb{R}$. Choose c such that the signal z has energy 1 and sketch z.
(e) Is the triangular pulse better aligned with the short or with the long pulse?

The final series of problems for this chapter deals with δ-functions as introduced in Section 2.5 and elaborated upon in Supplement C.

2.6.18. Products with δ-functions. Simplify the following δ-function expressions.

(a) $x(t) = \delta(t)e^{-|t|}$, $t \in \mathbb{R}$.
(b) $x(t) = \delta^{(2)}(t)e^{t^2}$, $t \in \mathbb{R}$.
(c) $x(t) = \delta^{(n)}(t)t^m$, $t \in \mathbb{R}$, with n and m nonnegative integers.
(d) $x(t) = \delta^{(n)}(t)\sin(t)$, $t \in \mathbb{R}$, with n a nonnegative integer.

In (c) and (d) we define $\delta^{(0)} = \delta$.

2.6.19. Symmetry property of derivatives of δ-functions. Prove that

$$\delta^{(n)}(-t) = (-1)^n\delta^{(n)}(t), \qquad t \in \mathbb{R},$$

for $n = 0, 1, 2, \cdots$, where $\delta^{(0)} = \delta$.

2.6.20. Differentiation of generalized signals. Determine the derivative Dx when x is given as follows.

(a) $x(t) = t^2\mathbb{1}(t)$, $t \in \mathbb{R}$.
(b) $x(t) = \mathrm{rect}(t)$, $t \in \mathbb{R}$.
(c) $x(t) = \sum_{n=-\infty}^{\infty} \mathbb{1}(t - nP)$, $t \in \mathbb{R}$, with P a positive real number.
(d) Determine all derivatives of the signal x given by

$$x(t) = \begin{cases} 0 & \text{for } t < 0, \\ t^N & \text{for } t \geq 0, \end{cases} \qquad t \in \mathbb{R},$$

with N a nonnegative integer.

2.6.21. Differentiation of a jump. Let x be a continuous-time signal that has a jump at time t_o, as shown in Fig. 2.40, but is otherwise continuously differentiable with derivative $\dot{x}$. Show that the derivative Dx of x is given by

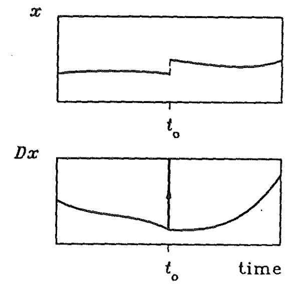


Figure 2.40 Differentiation of a jump. Top: signal with jump at time t_o. Bottom: its derivative.

$$Dx(t) = x'(t) + [x(t_o^+) - x(t_o^-)]\delta(t - t_o), \qquad t \in \mathbb{R},$$

where

$$x'(t) = \dot{x}(t) \quad \text{for} \quad t \in \mathbb{R} \quad \text{with } t \neq t_o,$$
$$x(t_o^+) = \lim_{\epsilon \downarrow 0} x(t_o + \epsilon),$$
$$x(t_o^-) = \lim_{\epsilon \downarrow 0} x(t_o - \epsilon).$$

As Fig. 2.40 shows, the derivative Dx has a δ-function at the time of the jump. The coefficient of the δ-function equals the size of the jump $x(t_o^+) - x(t_o^-)$.

2.6.22. Composite argument of the δ-function. Suppose that f: $\mathbb{R} \to \mathbb{R}$ is a continuously differentiable monotonically increasing function with derivative f' such that $f(\theta) = 0$ for some real θ. Prove that

$$\delta(f(t)) = \frac{1}{f'(\theta)}\delta(t - \theta), \qquad t \in \mathbb{R}.$$

2.6.23. Impact. An object with mass m moves along a straight line with constant speed v_o. At time 0 the object is subject to an impulse-like force given by

$$F(t) = P\delta(t), \qquad t \in \mathbb{R}.$$

How large should the momentum P be such that the motion of the object is stopped? *Hint:* If v is the speed of the object, by Newton's law $m dv(t)/dt = F(t)$.

2.7 COMPUTER EXERCISES

The computer exercises that follow may be carried out with the SIGSYS program. The exercises for Chapter 2 mainly serve to introduce the possibilities of SIGSYS to create and manipulate signals. In addition, the theory of this chapter is illustrated. Many of the exercises may also be done with other software, such as MATLAB.

2.7.1. Complex numbers. (Compare 2.6.1.) Given the following pairs of complex numbers x and y, compute their sum $x + y$, product xy, quotient x/y, as well as the exponential e^x. Represent the results both in Cartesian and in polar form.

(a) $x = 1 + j, y = 1 - j$. (c) $x = e^{j\pi/4}, y = 1 + j$.
(b) $x = e^{j\pi/2}, y = e^{-j\pi/2}$.

2.7.2. Various time signals. (Compare 2.6.6 and 2.6.7.) Plot the signal x for the following cases. In each case, choose the time axis with care.

(a) $x(t) = \mathbb{1}(t)\mathbb{1}(\theta - t)$, $t \in \mathbb{R}$, with $\theta = 1$.
(b) $x(t) = e^{-t}\mathbb{1}(t)$, $t \in \mathbb{R}$.
(c) $x(t) = \sin(\text{ramp}(t))$, $t \in \mathbb{R}$.
(d) $x(t) = \text{ramp}(\sin(t))$, $t \in \mathbb{R}$.

Representation of continous-time signals.
On a digital computer, continuous-time signals necessarily need be represented in sampled form. *Pointwise* operations, such as addition and multiplication, on continuous-time signals that are represented this way yield precise results but only at the sampling instants. Some other operations, however, such as integration, and the computation of the energy, action, and inner product, result in inaccuracies, because integrals are replaced with approximating sums.
When working with continuous-time signals, the values of the sampling interval (`inc` in SIGSYS) and the number of samples (`num` in SIGSYS) need be selected with what follows in mind:

1. Sufficient resolution (i.e., the sampling interval small enough and the number of samples large enough) to get good plots.
2. Sufficient resolution for numerical accuracy.
3. The length of the signal axis should be long enough to encompass all events of interest.
4. Choosing the number of samples large slows down computation and requires much memory.

Generating time signals.
In SIGSYS, time signals usually are most conveniently defined by means of signal composition with `dot` or `dotplus` as explained in Section 10 of the Tutorial. The time axis of the resulting signals is that of `dot` or `dotplus`, depending on which time axis is used. Before creating a time signal it is necessary to decide whether `dot` is used (for "two-sided" time axes) or `dotplus` (for "one-sided" time axes), and to select suitable values for the sampling interval `inc` and the number of sample points `num`.
Suppose by way of example that one wishes to plot one period of the signal x defined by

$$x(t) = \sin(2\pi t), \qquad t \in \mathbb{R}.$$

To get good resolution for plotting we take the number of samples equal to 100. Because the signal has period 1, we choose the sampling interval as $1/100 = 0.01$. By using a one-sided time axis, the time signal may be generated and displayed by typing

```
num=100
inc=1/num
x=sin(2*pi*dotplus)
plot x
```

(e) $x(t) = \text{rect}(t/2 - 1)$, $t \in \mathbb{R}$.
(f) $x(n) = a^n \mathbb{1}(n)$, $n \in \mathbb{Z}$, with $a = 0.9$.
(g) $x(n) = a^n \mathbb{1}(-n)$, $n \in \mathbb{Z}$, with $a = 1.1$.
(h) $x(n) = \sum_{k=0}^{\infty} \mathbb{1}(n - kN)$, $n \in \mathbb{Z}$, with $N = 4$.
(i) $x(n) = (-1)^n$, $n \in \mathbb{Z}$.
(j) $x(n) = j^n$, $n \in \mathbb{Z}$.

2.7.3. Periodic signals. Generate and plot several periods of the following periodic time signals.

(a) $x(t) = t \bmod 1$, $t \in \mathbb{R}$.
(b) $x(t) = \text{rect}(t \bmod 1)$, $t \in \mathbb{R}$.
(c) The discrete-time *infinite comb*, defined by

$$x(n) = \begin{cases} 1 & \text{for } n \bmod N = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad n \in \mathbb{Z},$$

with N a natural number. Take $N = 4$.

2.7.4. Discrete-time harmonic signals.

(a) *Aliasing.* Figure 2.9 shows that two real harmonic signals with frequencies $f_1 = 1/4$ and $f_2 = 5/4$ coincide on the time axis $\mathbb{T} = \mathbb{Z}$. Reproduce this plot.
(b) *A nonperiodic discrete-time harmonic signal.* According to 2.2.12, discrete-time harmonic signals on the time axis $\mathbb{Z}(T)$ are only periodic if their frequency is a rational multiple of $1/T$. To demonstrate this, plot the real harmonic signal c_f given by

$$c_f(t) = \cos(2\pi f t), \qquad t \in \mathbb{Z}(T),$$

with $f = \pi$ and $T = 1$ on the interval $[0, 50)$. The signal clearly is not periodic, but it seems to be close to periodic with period 7. Explain this. *Hint:* See 2.6.9.

2.7.5. Signal range transformation. (Compare 2.6.10.) In Fig. 2.11 it is shown what

(a) *full-wave rectification*

does to a real harmonic signal. Reproduce this figure. Show also what

(b) *half-wave rectification,*
(c) *hard limiting,*
(d) *soft limiting,* and
(e) *squaring*

do to the signal.

2.7.6. Time transformations. (Compare 2.6.11.) Let x be the signal given by

$$x(t) = \sin(2\pi f t), \qquad t \in \mathbb{R},$$

with f a given frequency.

(a) Choose a convenient time axis and a suitable value of f, and plot the signal.
(b) Plot the signal after time compression by a factor 2.

(c) Plot the signal after time expansion by a factor 2.
(d) Plot the signal after time reversal.
(e) Plot the signal after translating it by a nonzero time.

2.7.7. Some basic operations on signals.

(a) *Sampling and quantization.* Figure 2.17 shows the effect of sampling a continuous-time harmonic, and Fig. 2.13 demonstrates quantization. Reproduce these figures.
(b) *Mean, amplitude, rms value, maximum, minimum, sums, and products of signals.* In Figs. 2.6 and 2.7, plots are given of the rectangular and triangular pulse, the unit step, and the ramp signal. Reproduce these plots. Compute the mean, amplitude, rms value, maximum, and minimum of each of the four signals. Identify the effects of sampling and truncation. Also make plots of the sum and the product of the rectangular and triangular pulse and the sum and the product of the unit step and the ramp signal.

Truncation

Beware of the phenomenon called *truncation*. Some operations on signals involve signal values that lie *outside* the time axis on which the signal has been defined. This is usually resolved by replacing the missing values with zero. Inevitably this sometimes leads to incorrect results. Consider by way of example the problem of back shifting a unit step. The SIGSYS command

```
u=step(dot)
```

results in a unit step defined on a finite time axis. Shifting the result `u` according to

```
y=u(dot+10)
```

produces a signal `y` as in Fig. 2.41, which is zero near the end of the time axis. In this particular case the correct result may directly be obtained with the command

```
y=step(dot+10)
```

but this easy way out is not always available.

Another instance of truncation error arises when we compute the mean of the signal y by the SIGSYS command `mean(y)`. Even if y is correctly generated on the given finite time axis, by truncation this does not result in the correct value 0.5.

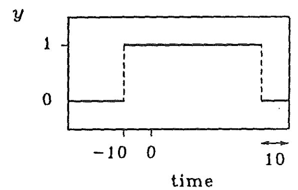


Figure 2.41 A back shifted truncated unit step.

2.7.8. Amplitude, energy and action. (Compare 2.6.13.) Plot the following signals and compute their amplitude, energy and action. Choose in each case the time axis with care. Identify the effects of truncation and sampling.

(a) $z(t) = \frac{1}{1+t}\mathbb{1}(t)$, $t \in \mathbb{R}$.
(b) $z(t) = \cos(2\pi f t)$, $t \in \mathbb{R}$, with $f = 1$.
(c) $z(n) = a^n \mathbb{1}(n)$, $n \in \mathbb{Z}$, with $a = 0.9$.
(d) $z(n) = a^{|n|}$, $n \in \mathbb{Z}$, with $a = 0.9$.

2.7.9. Correlation. In Example 2.4.11 it is shown how the inner product may be used for signal detection. If a signal is received that may be one of several signals, by taking inner products it may be found with which of those signals the received signal is best aligned. This method of signal detection has applications in radar and sonar. A frequent additional complication is that the received signal may have a *delay* so that its exact location in time is not known. To cope with this problem it is useful to introduce the *cross-correlation* $x \square y$ of two signals x and y, which is the signal given by

$$(x \square y)(n) = \sum_{k \in \mathbb{Z}} x(k+n)\overline{y(k)}, \qquad n \in \mathbb{Z},$$

if x and y are discrete-time signals, and by

$$(x \square y)(t) = \int_{-\infty}^{\infty} x(\tau + t)\overline{y(\tau)}\, d\tau, \qquad t \in \mathbb{R},$$

if x and y are continuous-time signals. For sampled signals defined on the time axis $\mathbb{Z}(T)$ we have

$$(x \square y)(t) = T \sum_{k \in \mathbb{Z}} x(kT + t)\overline{y(kT)}, \qquad t \in \mathbb{Z}(T).$$

The value of the cross-correlation $x \square y$ of x and y at time t simply is the inner product of x *back-shifted* by t, and y, that is,

$$(x \square y)(t) = \langle \sigma^t x, y \rangle.$$

Here, the back shift operator σ is defined by $(\sigma^\theta x)(t) = x(t + \theta)$, with t ranging over the appropriate time axis. By looking at the cross-correlation it may be determined for what time shift the signals x and y are best aligned.

By way of example, we again consider the situation of 2.4.11, where the received signal z is either the long pulse l or the short pulse s but is moreover *delayed* by an unknown time θ, that is, we have either

$$z(t) = l(t - \theta), \qquad t \in \mathbb{R},$$

or

$$z(t) = s(t - \theta), \qquad t \in \mathbb{R},$$

(a) Choose a suitable time axis, and generate the long pulse l and the short pul:
(b) Compute and plot the two cross-correlations $l \square l$ and $l \square s$. Verify that the m mum of the cross-correlation of l with l is 1 and that this occurs at time 0, that the maximum of the cross-correlation of l with s is $1/2$.
(c) Generate the signal z given by $z(t) = l(t - \theta)$, with $\theta = 5$, and compute cross-correlations $z \square l$ and $z \square s$. Verify that the maximum of the cross-corr tion of z and l is 1 and the maximum of the cross-correlation of z with s is 1 and that these maxima occur at time θ. Thus, it is reasonable to conclude that received signal z is the signal l delayed by θ.
(d) Generate the signal x given by

$$x(t) = \text{trian}(t/3 - 1), \qquad t \in \mathbb{R}.$$

Cross-correlate the signal x with l and s to see which signal it resembles mc and by how much it is delayed.
(e) To see not only which signal x resembles most but also *how much* it resembles i it is useful to scale x by a suitable factor such that its energy equals 1 (like that l and s). After doing this, what is the maximum of the cross-correlation of with l and s? Use the Cauchy-Schwarz inequality to prove that this maximum at most equal to 1.

2.7.10. Delta functions. Delta functions and their derivatives are essentially continuous time signals, but by choosing the sampling interval small enough they may be ap proximated by discrete-time signals. In what follows, choose the sampling interva small (e.g., equal to 0.01) and the time axis long enough (e.g., extending from -1 to 1) to encompass all events of interest.

(a) Define a signal u as the unit step, and differentiate it numerically to generate a signal d. See what this signal looks like.
(b) Numerically differentiate d twice more to obtain successively signals called d_1 and d_2. Plot them to see what they look like.
(c) The signal d is an approximation to the delta function δ, and d_1 and d_2 are approximations to the derivatives δ' and δ'', respectively. Define the signal ϕ as

$$\phi(t) = \begin{cases} 4(\frac{1}{4} - t^2) & \text{for } -\frac{1}{2} \le t < \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{R},$$

and check numerically in how far

$$\int_{-\infty}^{\infty} \delta^{(n)}(t)\phi(t)\, dt = (-1)^n \phi^{(n)}(0),$$

for $n = 0$, 1 and 2. Comment on the results. *Note:* $\delta^{(0)} = \delta$. *Hint:* Compute the integral as an inner product.

(d) Numerically differentiate the signal ϕ twice on the interval $[-1, 1]$. Explain what you find. *Hint:* Compare 2.6.21.

3

An Introduction to Systems

3.1 INTRODUCTION

As explained in Chapter 1, a *system,* very roughly, is a part of an environment that causes certain signals in that environment to be related. In this chapter we study *input-output systems,* abbreviated as *IO* systems. IO systems are driven by *input* signals and produce *output* signals. IO systems do not necessarily have a single possible output for each input. IO systems that are characterized by a *map* that assigns a *unique* output to each input are called *input-output mapping (IOM) systems*. *State* systems, which form another class of systems, are studied in Chapter 5.

In Section 3.2 the formal definitions of IO and IOM systems are given. Systems are categorized into *discrete-* and *continuous-time* systems, systems *with memory* and *memoryless* systems, *anticipating* and *non-anticipating* systems, and *time-varying* and *time-invariant* systems. We furthermore present examples of systems from electrical, mechanical, and chemical engineering, signal processing and economics. They are mainly "first-order." Higher-order systems are studied in Chapter 4.

Section 3.3 is devoted to *linear* systems. They derive their importance from the fact that many practical systems may be approximated by linear systems. In Section 3.4 we study *convolution systems*, which are IOM systems that are both linear and time-invariant. Such systems are relatively easy to analyze. In fact, most of this text is devoted to such systems. The convolution operation itself is extensively discussed in Section 3.5. An important notion that is introduced in Section 3.6 is that of the *stability* of convolution systems.

In Section 3.7 the response of convolution systems to harmonic inputs is considered, leading to a discussion of the *frequency response* of convolution systems. Section 3.8 deals with the response of convolution systems to *periodic* inputs, which results in the *cyclical convolution*. In Section 3.9, finally, it is shown how several systems may be *interconnected* to form a larger system.

3.2 INPUT-OUTPUT SYSTEMS AND INPUT-OUTPUT MAPPING SYSTEMS

Input-output systems are systems some of whose signals are designated as *inputs* and some as *outputs*. Usually, but not always, inputs are associated with *causes* and outputs with *effects*. A relationship or *rule* interrelates the input and output signals. Input-output systems are formally defined as follows.

3.2.1. Definition: Input-output systems. An *input-output* (IO) system is defined by a signal set $\mathcal{U}$, called the *input set*, a signal set $\mathcal{Y}$, called the *output set*, and a subset $\mathcal{R}$ of the product set $\mathcal{U} \times \mathcal{Y}$, called the *rule* or *relation* of the system. Any pair (u, y) with $u \in \mathcal{U}$, $y \in \mathcal{Y}$, and $(u, y) \in \mathcal{R}$ is said to be an *input-output pair* of the system, with u the *input signal* and y a corresponding *output signal*. ∎

Figure 3.1 illustrates the definition. It shows that an input-output system is simply defined as a collection $\mathcal{R}$ of input-output pairs (u, y). It also shows that for a given input u in general there is an entire set $\mathcal{Y}_u = \{y \in \mathcal{Y} \mid (u, y) \in \mathcal{R}\}$ of possible outputs that correspond to the input u. If for each input u there exists a *single* corresponding output y, then the system is said to be an *input-output mapping* (IOM) system.

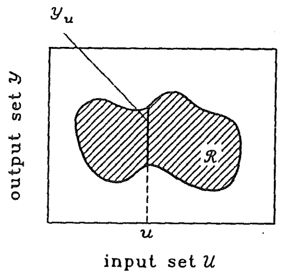


Figure 3.1 Input-output system: $\mathcal{R}$ is the set of possible input-output pairs $(u, y) \in \mathcal{U} \times \mathcal{Y}$.

3.2.2. Definition: Input-output mapping systems. Let $\mathcal{U}$ and $\mathcal{Y}$ be the input and output sets of an input-output system with rule $\mathcal{R} \subset \mathcal{U} \times \mathcal{Y}$. Then, if for each input u the fact that $(u, y_1) \in \mathcal{R}$ and $(u, y_2) \in \mathcal{R}$ implies that $y_1 = y_2$ the IO system is called an *input-output mapping* (IOM) system. The map $\phi: \mathcal{U} \to \mathcal{Y}$ that assigns a unique output $y \in \mathcal{Y}$ to each input $u \in \mathcal{U}$ is called the *input-output map* (IO map) of the system. ∎

Figure 3.2 illustrates the input-output relation of an IOM system. Both IO and IC systems are pictorially often represented by a block as in Fig. 3.3, with sometin the rule $\mathcal{R}$ written inside the block.

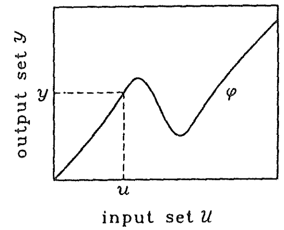


Figure 3.2 Input-output mapping system: For each input u there is a unique output $y = \phi(u)$.

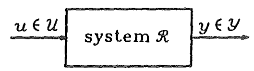


Figure 3.3 IO or IOM system in block diagram form.

Examples of IO and IOM Systems

We illustrate the definitions of IO and IOM systems with some examples.

3.2.3. Examples: IO and IOM systems.

(a) *Ball tossing system*. Given is a bag with a number of balls inside. Some more balls are tossed into the bag. The following signals may be associated with this system: the number of balls u that have been tossed in and the number of balls y that the bag now contains. Taking u as the input and y as the output, the input set $\mathcal{U}$ and the output set $\mathcal{Y}$ both consist of the set of nonnegative integers $\mathbb{Z}_+$. The rule of the system is

$$y \geq u.$$

In full set notation,

$$\mathcal{R} = \{(u, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid y \geq u\}.$$

Thus, (5, 6) is an IO pair but (6, 5) is not. The ball tossing system is an IO system, but not an IOM system, because to each u there correspond many, in fact infinitely many, possible outputs y.

If, on the other hand, it is known that the bag contained, say, y_o balls before tossing in additional balls, then the system is an *IOM* system, because the input u determines the output y uniquely by the formula

$$y = y_o + u.$$

This equation specifies the input-output map.

(b) *The exponential smoother and the RC network.* The exponential smoother of Example 1.2.6 and the RC network of Example 1.2.7 are input-output systems. In general they are not input-output mapping systems. However, if *fixed* initial conditions are specified once and for all (zero initial conditions, for instance), then the input-output relations of the systems become input-output maps, and consequently the systems are IOM systems. ■

Discrete- and Continuous-Time Systems

Most systems we study in this text have *time signals* as input and output. Such systems are called *dynamical* systems. The ball tossing system of Example 3.2.3(a) is not dynamical according to this definition, but the exponential smoother and the RC network are.

Usually, but not always, the input and output of a dynamical system have the *same* time axis $\mathbb{T}$. If both the input and the output of a dynamical system are discrete-time signals, then we call it a *discrete-time* system. If both the input and the output are continuous-time signals, then we speak of a *continuous-time* system. If one of the time axes is continuous and the other discrete, then the system is called a *hybrid* system.

3.2.4. Examples: Discrete-time, continuous-time, and hybrid systems. The exponential smoother of Example 1.2.6 is a discrete-time system. The RC network of Example 1.2.7 is a continuous-time system. The sampler of Fig. 2.16 is a hybrid system, because its input is a continuous-time signal and its output a discrete-time signal. ■

Memoryless IOM Systems

From Examples 1.2.6 and 1.2.7 it follows that $y(t)$, the output at time t, of the exponential smoother or the RC network depends on the *entire* past input u, and not just on the value $u(t)$ of the input at the same time t. There are also systems, called *memoryless* systems, where the current value of the output is fully determined by the current value of the input alone, and not by the past or future values of the input.

3.2.5. Definition: Memoryless system. An IOM system with time axis $\mathbb{T}$, input signal range U, and output signal range Y is memoryless if there exists a *pointwise* map ψ: $\mathbb{T} \times U \to Y$, such that if (u, y) is an input-output pair then

$$y(t) = \psi(t, u(t)), \qquad t \in \mathbb{T}.$$ ■

3.2.6. Examples: Memoryless systems.

(a) *Resistive circuit.* Consider the resistive circuit of Fig. 3.4, consisting of a voltage source connected to a resistive element with a voltage-current characteristic as in Fig. 3.5. We suppose that the input $u = v$ is a time-varying voltage. Hence,

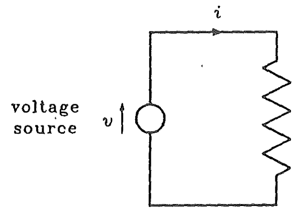


Figure 3.4 A resistive circuit.

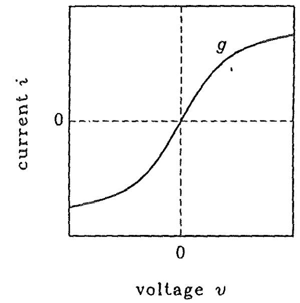


Figure 3.5 Voltage-current characteristic of the resistive element.

the time axis is $\mathbb{T} = \mathbb{R}$ and each input u is an element of the continuous-time signal space $\mathscr{L}$. The output $y = i$ is the time-varying current through the network and also an element of $\mathscr{L}$. If the voltage-current characteristic is represented by a function g as in the figure, at each time $t \in \mathbb{R}$ the output current $y(t)$ is uniquely determined by the input voltage $u(t)$ as

$$y(t) = g(u(t)), \quad t \in \mathbb{R}.$$

The resistive circuit therefore is an example of a memoryless system. If the voltage-current characteristic changes with time, for instance by ageing, the system still is memoryless, with an IO map of the form

$$y(t) = g(t, u(t)), \quad t \in \mathbb{R}.$$

(b) *Limiters, rectifiers, squarer.* Some well-known examples of memoryless systems follow:

(1) The *hard limiter* has the IO map

$$y(t) = \mathrm{sign}(u(t)), \qquad t \in \mathbb{T},$$

where sign: $\mathbb{R} \to \mathbb{R}$ is the *signum* or *sign* function defined by

$$\mathrm{sign}(u) = \begin{cases} 1 \text{ for } u > 0, \\ 0 \text{ for } u = 0, \\ -1 \text{ for } u < 0, \end{cases} \qquad u \in \mathbb{R}.$$

(2) The *soft limiter* has the IO map

$$y(t) = \mathrm{sat}(u(t)), \qquad t \in \mathbb{R},$$

where the *saturation function* sat: $\mathbb{R} \to \mathbb{R}$ is given by

$$\mathrm{sat}(u) = \begin{cases} 1 \text{ for } u > 1, \\ u \text{ for } |u| \leq 1, \\ -1 \text{ for } u < -1, \end{cases} \qquad u \in \mathbb{R}.$$

(3) The *half wave rectifier* is the memoryless system with IO map

$$y(t) = \mathrm{ramp}(u(t)), \qquad t \in \mathbb{R},$$

where ramp: $\mathbb{R} \to \mathbb{R}$ is the function defined by

$$\mathrm{ramp}(u) = \begin{cases} 0 \text{ for } u < 0, \\ u \text{ for } u \geq 0, \end{cases} \qquad u \in \mathbb{R}.$$

Figure 3.6 shows the graphs of the sign, sat, and ramp functions.

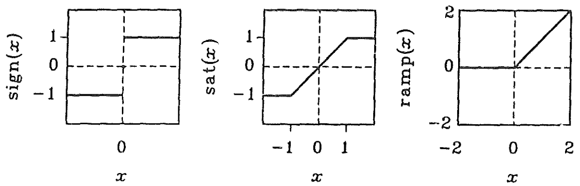


Figure 3.6 Left: the signum function. Middle: the saturation function. Right: the ramp function.

(4) The *full wave rectifier* is the memoryless system with IO map

$$y(t) = |u(t)|, \qquad t \in \mathbb{T}.$$

(5) The *squarer*, finally, is the memoryless system described by

$$y(t) = [u(t)]^2, \qquad t \in \mathbb{T}.$$

■

Non-Anticipating Systems

Real-life systems have the property that their output $y(t)$ at any given time t only depends on the values of the input *before* time t, and not on later values of the input. This property is called *non-anticipativeness*. Theoretically, we may, of course, well hypothesize systems whose current output does not only depend on past values of the input but also on *future* values. Such systems are said to be *anticipative* or *anticipating*.

Formally, non-anticipativeness of IOM systems is defined as follows. In this definition, the system is defined on the time axis $\mathbb{T}$. The input signal has signal range U, so that the input set consists of all time signals $\mathcal{U} = \{u: \mathbb{T} \to U\}$. Likewise, the output signal has signal range Y and hence belongs to the output set $\mathcal{Y} = \{y: \mathbb{T} \to Y\}$.

3.2.7. Definition: Non-anticipating dynamical IOM systems. Consider an IOM system with time axis $\mathbb{T}$ and rule $\mathcal{R} \subset \mathcal{U} \times \mathcal{Y}$. Let (u_1, y_1) be any IO pair and $t \in \mathbb{T}$ an arbitrary time, and suppose that (u_2, y_2) is any other input-output pair such that $u_1(\tau) = u_2(\tau)$ for all $\tau \leq t$ with $\tau \in \mathbb{T}$. Then the system is *non-anticipating* if $y_1(\tau) = y_2(\tau)$ for all $\tau \leq t$ such that $\tau \in \mathbb{T}$. ■

What follows are examples of systems that may or may not be anticipating.

3.2.8. Examples: Anticipativeness and non-anticipativeness.

(a) *Pure delay and pure predictor*. Consider the discrete-time or continuous-time IOM system whose time axis $\mathbb{T}$ is either $\mathbb{Z}$ or $\mathbb{R}$ and whose output is given by

$$y(t) = u(t - \theta), \qquad t \in \mathbb{T},$$

with $\theta \in \mathbb{T}$ fixed. Figure 3.7 shows that if $\theta \geq 0$ the system *delays* the input by θ, which is why it is called a *pure delay*. The pure delay is non-anticipating. If $\theta < 0$ the system is called a *pure predictor*. Predictors are obviously anticipating.

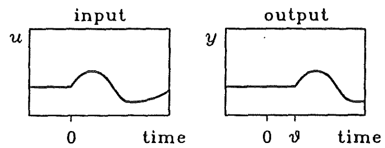


Figure 3.7 IO pair of a pure delay system. Left: input. Right: output.

(b) *Discrete-time sliding window averager*. The rule

$$y(n) = \frac{1}{N + M + 1} \sum_{m=-M}^{N} u(n + m), \qquad n \in \mathbb{Z},$$

with N and M nonnegative integers, describes a discrete-time IOM system, called *sliding window averager*. The system takes an arithmetic average of the input, with a "window" extending from M time intervals before until N intervals after the time n at which the average is taken. The system is non-anticipating if and only if $N = 0$.

(c) *Continuous-time sliding window averager*. The continuous-time equivalent of the discrete-time sliding window averager is the system with rule

$$y(t) = \frac{1}{T_1 + T_2} \int_{t-T_1}^{t+T_2} u(\tau)\, d\tau, \qquad t \in \mathbb{R},$$

with T_1 and T_2 nonnegative real numbers such that $T_1 + T_2 > 0$ (Fig. 3.8.) The system is non-anticipating if and only if $T_2 = 0$.

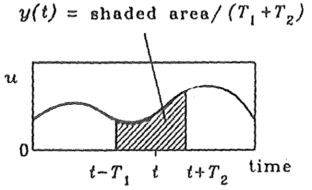


Figure 3.8 Continuous-time sliding window averaging. ∎

Time-Invariance

We next discuss the notion of *time-invariance*. A system is time-invariant, roughly, if the system properties do not change with time. Most of the systems studied in this text are time-invariant. To define time-invariance, we introduce the *back shift operator* σ^θ, with θ a real number, which shifts a time signal *backward* if $\theta > 0$ and forward if $\theta < 0$.

3.2.9. Definition: Back shift operator. Let x be a discrete- or continuous-time signal defined on an infinite or semi-infinite time axis $\mathbb{T}$. Let θ be a real number such that $t + \theta \in \mathbb{T}$ for all $t \in \mathbb{T}$. Then, the *back shift operator* σ^θ maps the signal x to the back shifted signal $\sigma^\theta x$ given by

$$(\sigma^\theta x)(t) = x(t + \theta), \qquad t \in \mathbb{T}. \quad ∎$$

Figure 3.9 illustrates the back shift operator. Note that the admissible time shifts θ depend on the time axis. If $\mathbb{T}$ is an infinite time axis such as $\mathbb{Z}$ or $\mathbb{R}$, θ may take any value in $\mathbb{T}$. If $\mathbb{T}$ is right semi-infinite such as $\mathbb{Z}_+$, $\mathbb{R}_+$, or $[t_0, \infty)$, then only *nonnegative* values of θ in $\mathbb{T}$ are allowed.

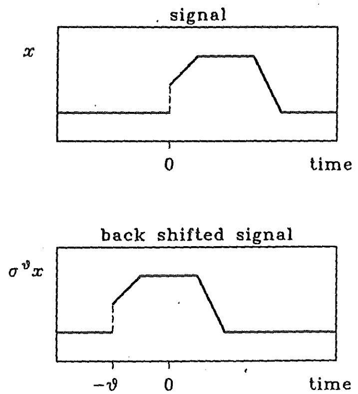


Figure 3.9 Back shift operator. Top: a time signal with time axis $\mathbb{R}$. Bottom: the same time signal after application of the back shift operator with positive parameter θ.

As was noted before, a system is time-invariant if its properties do not change with time: Any input-output pair (u, y) may be arbitrarily shifted in time and *remains* an IO pair. Time-varying systems do not have this property. The formal definition of time-invariance is as follows.

3.2.10. Definition: Time-invariance. Consider an IO system with the infinite or right semi-infinite discrete or continuous time axis $\mathbb{T}$. Then the system is *time-invariant* if for every input-output pair (u, y) also the time-shifted pair $(\sigma^\theta u, \sigma^\theta y)$ is an input-output pair for any allowable time shift θ. ∎

For IOM systems with IO map ϕ, time-invariance reduces to *shift-invariance* of the IO map ϕ, that is,

$$\phi(\sigma^\theta u) = \sigma^\theta \phi(u)$$

for every input u and every allowable time shift θ. Shift-invariance of the IO map is equivalent to the statement that ϕ and σ^θ commute for all θ, as illustrated in Fig. 3.10.

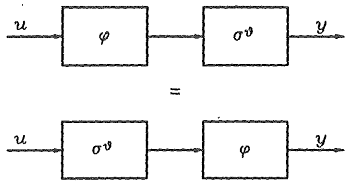


Figure 3.10 Shift invariance.

A system that is not time-invariant is said to be *time-varying*. We illustrate time-invariance with some examples.

3.2.11. Examples: Time-invariant and time-varying systems.

(a) *The RC network*. The RC network of Example 1.2.7 is a continuous-time system with input u and output y, described by the differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}u(t), \qquad t \in \mathbb{R}, \tag{1}$$

where we study the network on the infinite time axis $\mathbb{R}$. Suppose that (u, y) is an IO pair, satisfying the differential equation. The system is time-invariant if we can prove that the back shifted pair (u', y'), with

$$u' = \sigma^\theta u, \qquad y' = \sigma^\theta y,$$

also is an IO pair, i.e., also satisfies the differential equation, for every real θ. Indeed, using the chain rule for differentiation it follows by differentiation with respect to t and using (1) that

$$\frac{d}{dt}\sigma^\theta y(t) = \frac{d}{dt}y(t+\theta) = \frac{dy}{dt}(t+\theta) = -\frac{1}{RC}y(t+\theta) + \frac{1}{RC}u(t+\theta)$$

$$= -\frac{1}{RC}(\sigma^\theta y)(t) + \frac{1}{RC}(\sigma^\theta u)(t), \qquad t \in \mathbb{R},$$

which shows that (u', y') is an IO pair.

Suppose now that the resistance of the resistor is not constant, but *varies* with time, perhaps by ageing or as a result of changing environmental temperature. The differential equation (1) then takes the form

$$\frac{dy(t)}{dt} + \frac{1}{R(t)C}y(t) = \frac{1}{R(t)C}u(t), \qquad t \in \mathbb{R},$$

where $R(t)$ is the value of the resistance at time t. We assume that $R(t) > 0$ for all t. Let us check whether this IO system is time-invariant. The back shifted pair (u', y') is a solution of the differential equation if

$$\frac{d}{dt}y(t+\theta) + \frac{1}{R(t)C}y(t+\theta) = \frac{1}{R(t)C}u(t+\theta), \qquad t \in \mathbb{R}.$$

Replacing t with $\tau - \theta$ we thus require that

$$\frac{dy(\tau)}{d\tau} + \frac{1}{R(\tau-\theta)C}y(\tau) = \frac{1}{R(\tau-\theta)C}u(\tau), \qquad \tau \in \mathbb{R}.$$

Even though (u, y) is an IO pair this equation is not necessarily satisfied, unless $R(\tau - \theta) = R(\tau)$ for all $\tau \in \mathbb{R}$ and for any $\theta \in \mathbb{R}$, which implies that the resistance is constant. Thus, if the resistance is not constant, a backshifted IO pair in general is *not* an IO pair, and, as a result, the system is *time-varying*.

(b) *Time-invariance of a memoryless IOM system.* A memoryless IOM system with pointwise IO map given by

$$y(t) = \psi(t, u(t)), \qquad t \in \mathbb{T},$$

is time-invariant if and only if $\psi(t_1, u) = \psi(t_2, u)$ for all t_1 and t_2 on the time axis $\mathbb{T}$, and all u belonging to the input signal range of the system. Thus, a memoryless system is time-invariant if and only if the function ψ does not depend on its first argument, which it time. The soft and hard limiter, the half- and full-wave rectifier, and the squarer of Example 3.2.6(b) are all time-invariant memoryless systems. ∎

Additional Examples of Input-Output Systems

In conclusion of this section we present several further examples of input-output systems originating from different areas.

3.2.12. Example: Heated vessel. Figure 3.11 shows a vessel containing continuously stirred fluid, which is heated by an electrical coil. The input to the system is the electrical power u dissipated by the coil, while its output is the temperature T of the fluid. We assume that the vessel is stirred well, so that the fluid in the vessel has the same temperature everywhere. The heat loss to the environment is proportional to the difference $T - T_e$ between the temperature of the fluid and the environment temperature T_e, which is assumed to be constant. Setting up the heat balance of the system we obtain the equation

$$C\frac{dT(t)}{dt} = u(t) - \frac{T(t) - T_e(t)}{R}, \qquad t \in \mathbb{R},$$

where C is the heat capacity of the fluid and R the heat resistance to the environment. The term on the left-hand side is the heat flow needed to increase the temperature of the fluid, the first term on the right-hand side is the external supply of heat, and the second term the loss of heat to the environment. Redefining the output as the temperature *difference* $y = T - T_e$, and substituting $T = y + T_e$ into the differential equation, we easily find that

$$\frac{dy(t)}{dt} = -\frac{1}{RC}y(t) + \frac{1}{C}u(t), \qquad t \in \mathbb{R}.$$

This differential equation is similar to that for the RC circuit of Example 1.2.7. It describes the heated vessel as a continuous-time IO system.

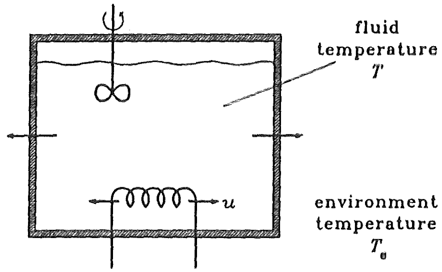


Figure 3.11 A heated vessel. ∎

3.2.13. Example: The motion of a car. In the final part of Section 1.3 we discussed a cruise control system for a car. The motion of the car (without the cruise control system) may be described by the following simple model, which accounts for the major physical effects. By Newton's law,

$$M\frac{dv(t)}{dt} = F_{\text{total}}(t), \qquad t \geq t_o,$$

where M is the mass of the car, dv/dt its acceleration, and F_{total} the total force exerted on the car in forward direction. The total force may be expressed as

$$F_{\text{total}}(t) = cu(t) - Bv^2(t).$$

The first term $cu(t)$ represents the pulling force of the engine, and is proportional to the throttle opening $u(t)$, with proportionality constant c. The second term $Bv^2(t)$ is caused by air resistance. This friction force is proportional to the square of the car speed v, with B representing the friction coefficient. Substitution of F_{total} into Newton's law results in

$$M\frac{dv(t)}{dt} = cu(t) - Bv^2(t), \qquad t \geq t_o.$$

All input-output pairs (u, v) need satisfy this nonlinear first-order differential equation. The system is a continuous-time IO system. ■

3.2.14. Example: Savings account. Suppose that $y(n)$ represents the balance of a savings account at the beginning of day n, and $u(n)$ the amount deposited during day n. If interest is computed and added daily at a rate of $\alpha \cdot 100\%$, the balance at the beginning of day $n + 1$ is given by

$$y(n + 1) = (1 + \alpha)y(n) + u(n), \qquad n = 0, 1, 2, \cdots.$$

This describes the savings account as a discrete-time system on the time axis $\mathbb{Z}_+$. It is an IO system. If the interest rate α does not change with time, the system is time-invariant; otherwise, it is time-varying. ■

3.2.15. Example: National economy. A very crude model for the national economy of a country follows. Let $y(n)$ represent the total capital stock of the country accumulated at the beginning of year n. During the year a fraction r of the capital stock is lost owing to ageing and obsolescence. The production during the year is proportional, with productivity coefficient p, to the capital stock. The aggregate consumption during the year is $u(n)$ and is considered the input to the system. The relation

$$y(n + 1) = y(n) - ry(n) + py(n) - u(n), \qquad n = 0, 1, 2, \cdots,$$

expresses that the capital stock at the beginning of the next year equals that of the present year, decreased by the amounts $ry(n)$ lost by depreciation and $u(n)$ used for consumption, and increased by the production $py(n)$. The equation may be rewritten as

$$y(n + 1) = (1 + p - r)y(n) - u(n), \qquad n = 0, 1, 2, \cdots.$$

This difference equation describes the national economy as a discrete-time IO system with time axis $\mathbb{Z}_+$. If the constants p and r do not change with time, the system is time-invariant. ■

The four systems presented in 3.2.12 to 3.2.15 are all input-output systems but not input output mapping systems. In each case the output y does not depend exclusively on the input but also on the initial value of the output.

3.3 LINEAR SYSTEMS

Linear systems form an important class of systems, because, on the one hand, they lend themselves well to mathematical analysis while, on the other, many practical engineering systems may be accurately modeled as linear systems.

3.3.1. Definition: Linear IO system. Let $\mathcal{U}$ and $\mathcal{Y}$ be the input and output sets, respectively, of an input-output system with rule $\mathcal{R} \subset \mathcal{U} \times \mathcal{Y}$. The IO system is *linear* if $\mathcal{U}$ and $\mathcal{Y}$ are *linear spaces* and $\mathcal{R}$ is a *subspace* of $\mathcal{U} \times \mathcal{Y}$. ■

By definition, R is a subspace if it has the property that if (u_1, y_1) and (u_2, y_2) are any two IO pairs, the linear combination

$$(\alpha u_1 + \beta u_2, \alpha y_1 + \beta y_2)$$

is also an IO pair for arbitrary scalars α and β. The scalars usually take values in $\mathbb{R}$ or $\mathbb{C}$, depending on the field over which the spaces $\mathcal{U}$ and $\mathcal{Y}$ are linear.

An IO system that is *not* linear is said to be *nonlinear*.

3.3.2. Example: Linearity of the exponential smoother. The exponential smoother, which in Example 1.2.6 is described as an IO system on the time axis $\{0, 1, 2, \cdots\}$, is linear. The reason is that first of all the input and output sets, consisting of all complex-valued sequences on the semi-infinite time axis $\mathbb{Z}_+$, are linear spaces over the complex numbers. To show that the rule $\mathcal{R}$ is linear let (u_1, y_1) and (u_2, y_2) be two input-output pairs, that is, they satisfy

$$y_1(n + 1) = ay_1(n) + (1 - a)u_1(n + 1), \qquad n = 0, 1, 2, \cdots,$$
$$y_2(n + 1) = ay_2(n) + (1 - a)u_2(n + 1), \qquad n = 0, 1, 2, \cdots.$$

Multiplying the first of these equations by the complex scalar α, the second by β, and adding the two resulting expressions yields

$$[\alpha y_1(n+1) + \beta y_2(n+1)] = a[\alpha y_1(n) + \beta y_2(n)] + (1-a)[\alpha u_1(n+1) + \beta u_2(n+1)],$$

$n = 0, 1, 2, \cdots$. This proves that $(\alpha u_1 + \beta u_2, \alpha y_1 + \beta y_2)$ is an IO pair. Hence, $\mathcal{R}$ is a subspace and the system is linear. ∎

3.3.3. Exercise: Linearity of the RC network. Prove that the RC circuit of Example 1.2.7, with unspecified initial conditions, is a linear IO system on the time axis $[t_o, \infty)$. Show that it is also linear if the resistor is time-varying, as at the end of Example 3.2.11(a). ∎

Linearity of IOM Systems

Linearity of an input-output mapping system is equivalent to linearity of its IO map.

3.3.4. Summary: Linearity of IOM systems. The IOM system with input set $\mathcal{U}$, output set $\mathcal{Y}$ and IO map $\phi: \mathcal{U} \to \mathcal{Y}$ is linear if and only if $\mathcal{U}$ and $\mathcal{Y}$ are linear spaces, and ϕ is a *linear map,* that is,

$$\phi(\alpha u_1 + \beta u_2) = \alpha\phi(u_1) + \beta\phi(u_2) \tag{1}$$

for every u_1 and u_2 in $\mathcal{U}$ and all scalars α and β. ∎

The property (1) is known as the *superposition* property of linear maps. A necessary and sufficient condition for the map ϕ to be linear is that it be both *additive,* that is,

$$\phi(u_1 + u_2) = \phi(u_1) + \phi(u_2)$$

for every u_1 and u_2 in $\mathcal{U}$, and *homogeneous,* that is,

$$\phi(\alpha u) = \alpha\phi(u)$$

for every scalar α and each $u \in \mathcal{U}$. The next exercise shows that both additivity and homogeneity are needed for linearity.

3.3.5. Exercise: Homogeneity, additivity, and linearity.

(a) Show that additivity by itself does not imply linearity. *Hint:* Take as a counterexample $U = Y = \mathbb{C}$ and ϕ defined by $\phi(u) = \mathrm{Re}(u)$, and consider linearity over the complex numbers.

(b) Show that also homogeneity by itself does not imply linearity. *Hint:* Take as a counterexample $\mathcal{U} = \mathbb{R} \times \mathbb{R}$, $\mathcal{Y} = \mathbb{R}$ and ϕ defined by $\phi(u_1, u_2) = (u_1^3 + u_2^3)^{1/3}$, and consider linearity over the real numbers. ∎

We present some examples of linear and nonlinear IOM systems.

3.3.6. Example: Linearity of the sliding window averager. The IO map (the discrete-time sliding window averager of Example 3.2.8(b) is described by th relation

$$y(n) = \frac{1}{N+M+1}\sum_{k=-M}^{N} u(n+k), \qquad n \in \mathbb{Z}.$$

By the linearity of the summing operation this IO map is linear, and, hence, the discrete-time sliding window averager is a linear IOM system.

Also the continuous-time sliding window averager of Example 3.2.8(c) is a linear IOM system. ∎

3.3.7. Example: Two nonlinear IOM systems.

(a) *Ball tossing system.* The ball tossing system of Example 3.2.3(a) is nonlinear because the input and output sets $\mathbb{Z}_+$ are not linear spaces.

(b) *Memoryless nonlinear systems.* The memoryless resistive circuit of Example 3.2.6(a) is nonlinear *unless* the voltage-current characteristic is a straight line crossing through the origin, that is, the function g is of the form

$$g(v) = v/R,$$

with R a constant.

The hard and soft limiter, the half- and full-wave rectifier, and the squarer of Example 3.2.6(b) are all nonlinear. ∎

Linearization

Many practical IO systems may be approximated, at least "locally," by a linear IO system. The approximation procedure is called *linearization.* Linearization is an extension of the idea that a curve may be approximated in the neighborhood of a point on the curve by the tangent at the point, as illustrated in Fig. 3.12.

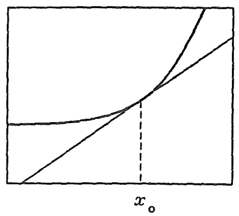


Figure 3.12 Approximation of a curve by its tangent at a point x_o.

It is important to note the three main features of linearization:

(i) Linearization always takes place about some fixed "operating point" (the point x_o in Fig. 3.12).
(ii) The approximation is only valid if the system behaves "smoothly" near the operating point.
(iii) The approximation only holds for "small" deviations from the operating point.

We demonstrate linearization of systems with two examples.

3.3.8. Example: Linearization of the resistive circuit. The resistive circuit of Example 3.2.6(a) generally is a nonlinear IOM system, described by the IO map

$$i = g(v).$$

Let (v_o, i_o) be a point on the voltage-current characteristic, (i.e., $i_o = g(v_o)$). Any other input voltage v and the corresponding output current i may be written as

$$v = v_o + \tilde{v}, \qquad i = i_o + \tilde{i},$$

with $\tilde{v}$ and $\tilde{i}$ the deviations of the voltage and current, respectively, from the operating point values. It follows that

$$i_o + \tilde{i} = g(v_o + \tilde{v}),$$

as shown in Fig. 3.13. If $\tilde{v}$ is small and g is differentiable at the point v_o with derivative $g'(v_o)$, by using the first term of the Taylor expansion of the right-hand side we may approximate as

$$i_o + \tilde{i} \approx g(v_o) + g'(v_o)\tilde{v}.$$

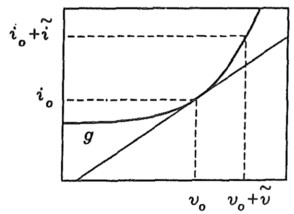


Figure 3.13 Linearization of the voltage-current characteristic of the resistive element at the point (v_o, i_o).

Since $i_o = g(v_o)$, the first terms on both sides of the expression cancel and we have

$$\tilde{i} \approx g'(v_o)\tilde{v}.$$

Let us denote the deviation of the voltage $u = \tilde{v}$ as the input of the system, and the deviation of the current $y = \tilde{i}$ as its output. Then, replacing the approximate equality with equality, we obtain the IO map

$$y = \frac{u}{R_o},$$

where $R_o = 1/g'(v_o)$. This *linear* IO map describes the *linearized* or *variational* system about the operating point (v_o, i_o). R_o is the *equivalent resistance* of the nonlinear resistive element at the operating point. In particular, if, for instance, $g(v) = \alpha v + \beta v^3$, then $g'(v_o) = \alpha + 3\beta v_o^2$ and

$$R_o = \frac{1}{\alpha + 3\beta v_o^2}.$$ ∎

3.3.9. Example: Linearization of the differential equation for the motion of a car. In Example 3.2.13 we saw that the motion of a car may be described by the differential equation

$$M\frac{dv(t)}{dt} = cu(t) - Bv^2(t), \qquad t \geq t_o,$$

with the input u the throttle position and the output v the car speed. It is simple to verify that if (u_1, v_1) is any solution pair to the differential equation and (u_2, v_2) another, an arbitrary linear combination $(\alpha u_1 + \beta u_2, \alpha v_1 + \beta v_2)$, with α and β real, in general does *not* satisfy the differential equation. The culprit is the quadratic term in the differential equation, and the result is that the IO system is nonlinear.

To linearize, we first choose an "operating point," which in principle may be any IO pair (u_o, y_o) satisfying the differential equation. A convenient choice is to take for u_o the constant signal $u_o(t) = U_o$, and for v_o the corresponding constant solution $v_o(t) = V_o$ of the differential equation. Substitution into the equation yields $0 = cU_o - BV_o^2$, so that $V_o = \sqrt{cU_o/B}$. The constant input U_o, of course, represents a constant throttle position, and V_o is the constant cruising speed corresponding to this throttle position.

The next step is to write

$$u(t) = U_o + \tilde{u}(t), \qquad v(t) = V_o + \tilde{v}(t), \qquad t \geq t_o,$$

with $\tilde{u}$ and $\tilde{v}$ the deviations of u and v from U_o and V_o, respectively. Substitution into the differential equation yields

$$M\frac{d}{dt}[V_o + \tilde{v}(t)] = c[U_o + \tilde{u}(t)] - B[V_o + \tilde{v}(t)]^2, \qquad t \geq t_o.$$

By using the equality $cU_o = BV_o^2$, this reduces to

$$M\frac{d}{dt}\tilde{v}(t) = c\tilde{u}(t) - 2BV_o\tilde{v}(t) - B\tilde{v}^2(t), \qquad t \geq t_o.$$

If $\tilde{v}$ is small, we may neglect the quadratic term on the right-hand side and obtain the *linear* differential equation

$$M\frac{d}{dt}\tilde{v}(t) = c\tilde{u}(t) - 2BV_o\tilde{v}(t), \qquad t \geq t_o. \tag{2}$$

We may now redefine the input and output to be $\tilde{u}$ and $\tilde{v}$, respectively, and call the IO system described by (2) the *linearized* or *variational* system about the solution (u_o, v_o). It is easy to verify that the variational system is linear because the differential equation (2) is linear.

By using the variational system we may obtain approximations to the original nonlinear system that are quite good as long as $\tilde{v}$ is small. ■

The IO Map of Linear IOM Systems

We continue by showing that the IO map of linear dynamical IOM systems may be expressed explicitly in the form of a sum (in the discrete-time case) or an integral (in the continuous-time case.)

Consider first a discrete-time IOM system whose IO map may be represented as

$$y(n) = \sum_{m=-\infty}^{\infty} k(n, m)\, u(m), \qquad n \in \mathbb{Z}, \tag{3}$$

with k a given function of two variables and u and y real- or complex-valued signals. The function k is called the *kernel* of the system. An example of a system whose IO map takes this form is the discrete-time sliding window averager of Example 3.2.8(b), where

$$k(n, m) = \begin{cases} \dfrac{1}{N + M + 1} & \text{for } n - M \leq m \leq n + N, \\ 0 & \text{otherwise,} \end{cases} \qquad n, m \in \mathbb{Z}.$$

It is easy to show that the IO map (3) represents a *linear* system because it has the superposition property.

The kernel k has the following interpretation. Suppose that the input u to the system described by (3) is

$$u(n) = \Delta(n - n_o), \qquad n \in \mathbb{Z},$$

that is, the input u is the unit pulse shifted to the time $n_o \in \mathbb{Z}$. Then, from (3) corresponding output is

$$\begin{aligned} y(n) &= \sum_{m=-\infty}^{\infty} k(n, m)u(m) = \sum_{m=-\infty}^{\infty} k(n, m)\Delta(m - n_o) \\ &= k(n, n_o), \qquad n \in \mathbb{Z}. \end{aligned}$$

Thus, the value $k(n, m)$ of k at the point (n, m) is the *response of the system at tin n if the input is a unit pulse applied at time m.* Figure 3.14 illustrates this for an arb trary system.

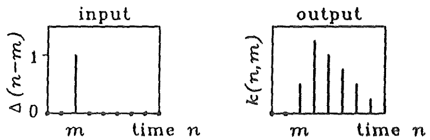


Figure 3.14 Kernel of a discrete-time linear system. Left: unit pulse at time m Right: the response is $k(n, m)$, $n \in \mathbb{Z}$.

Similar to the discrete-time case, continuous time IOM systems whose IO map may be represented in the form

$$y(t) = \int_{-\infty}^{\infty} k(t, \tau)u(\tau)\, d\tau, \qquad t \in \mathbb{R}, \tag{4}$$

are linear. The function k is again called the *kernel* of the system. The continuous-time sliding window averager is such a system, with kernel

$$k(t, \tau) = \begin{cases} \dfrac{1}{T_1 + T_2} & \text{for } t - T_1 \leq \tau < t + T_2, \\ 0 & \text{otherwise,} \end{cases} \qquad t, \tau \in \mathbb{R}.$$

Suppose that the input u to the system described by (4) is a δ-function shifted to the time τ_o, that is,

$$u(\tau) = \delta(\tau - \tau_o), \qquad \tau \in \mathbb{R}.$$

Then the output is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} k(t, \tau)u(\tau)\, d\tau = \int_{-\infty}^{\infty} k(t, \tau)\delta(\tau - \tau_o)\, d\tau \\ &= k(t, \tau_o), \qquad t \in \mathbb{R}. \end{aligned}$$

Thus, $k(t, \tau)$ is the response at time t when the input is a delta function shifted to time τ.

So far we have seen that an IO map given in the form (3) or (4) defines a linear IOM system. What is perhaps surprising is that the IO map of *every* linear IOM system on the infinite time axis $\mathbb{Z}$ or $\mathbb{R}$ may be written in the form (3) or (4). We first state the complete result.

3.3.10. Summary: The IO map of linear IOM systems.

A discrete-time IOM system with time axis $\mathbb{Z}$ and real- or complex-valued input and output is linear if and only if its IO map is of the form

$$y(n) = \sum_{m=-\infty}^{\infty} k(n, m)u(m),$$

$n \in \mathbb{Z}$. The function k, called the *kernel* of the system, is any function $\mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$.

A continuous-time IOM system with time axis $\mathbb{R}$ and real- or complex-valued input and output is linear if and only if its IO map is of the form

$$y(t) = \int_{-\infty}^{\infty} k(t, \tau)u(\tau)\, d\tau,$$

$t \in \mathbb{R}$. The function k, called the *kernel* of the system, is any function $\mathbb{R} \times \mathbb{R} \to \mathbb{C}$, possibly singular. ∎

To see that any linear IOM system may be represented as in 3.3.10, first consider an arbitrary discrete-time linear IOM system and denote the response of the system at time n to a unit pulse at time m as $k(n, m)$. An arbitrary input u may be represented as a linear combination of shifted unit pulses in the form

$$u(n) = \sum_{m=-\infty}^{\infty} u(m)\Delta(n - m), \qquad n \in \mathbb{Z}. \tag{5}$$

By *homogeneity* the response of the system to the shifted and scaled pulse $u(m)\Delta(n - m)$, $n \in \mathbb{Z}$, is $k(n, m)u(m)$, $n \in \mathbb{Z}$, and by *additivity* the response to the composite input (5) is

$$y(n) = \sum_{m=-\infty}^{\infty} k(n, m)u(m), \qquad n \in \mathbb{Z}.$$

This proves that the IO map of *every* linear discrete-time IOM system may be represented in the form (3).

A similar argument holds for continuous-time linear IOM systems. Denote the response of the system to the shifted δ-function $\delta(t - \tau)$, $t \in \mathbb{R}$, as $k(t, \tau)$, $t \in \mathbb{R}$. Any input u to the system may be decomposed as an infinite linear combination of shifted δ-functions of the form

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t - \tau)\, d\tau, \qquad t \in \mathbb{R}.$$

By homogeneity and addivity it is plausible that the response to this input is

$$y(t) = \int_{-\infty}^{\infty} k(t, \tau)u(\tau)\, d\tau, \qquad t \in \mathbb{R}.$$

The discrete- and continuous-time sliding window averager were already mentioned as examples of systems whose IO map may be represented in sum or integral form. A further example is the following.

3.3.11. Example: The kernel of a delay. The pure delay of Example 3.2.8(a) is characterized by

$$y(t) = u(t - \theta), \qquad t \in \mathbb{T},$$

with θ the delay.

(a) *Discrete-time case.* When the time axis is $\mathbb{T} = \mathbb{Z}$, the system is discrete-time. The response of the discrete-time delay to the shifted unit pulse $u(n) = \Delta(n - m)$, $n \in \mathbb{Z}$, is

$$y(n) = \Delta(n - \theta - m), \qquad n \in \mathbb{Z}.$$

Hence, the kernel of the system is

$$k(n, m) = \Delta(n - \theta - m) = \begin{cases} 1 & \text{for } n = m + \theta, \\ 0 & \text{otherwise,} \end{cases} \qquad n, m \in \mathbb{Z}.$$

(b) *Continuous-time case.* If $\mathbb{T} = \mathbb{R}$ the system is continuous-time. The response to the shifted delta function $u(t) = \delta(t - \tau)$, $t \in \mathbb{R}$, then is

$$y(t) = \delta(t - \theta - \tau), \qquad t \in \mathbb{R}.$$

As a result, the kernel of the system is

$$k(t, \tau) = \delta(t - \theta - \tau), \qquad t, \tau \in \mathbb{R}.$$

Thus, the continuous-time delay system has a *singular* kernel. ∎

Non-Anticipating and Real Linear IOM Systems

In the discrete-time case the interpretation of the kernel k of a linear IOM system is that $k(n, m)$ is the response of the system at time n to a unit pulse at time m. In the continuous-time case $k(t, \tau)$ is the response at time t to a delta function at time τ. A non-anticipating system clearly cannot respond to a pulse or delta function before its arrival. Thus, a necessary condition for non-anticipativeness is both in the discrete- and the continuous-time case that $k(t, \tau) = 0$ for $t < \tau$. It is easy to see that this is also a sufficient condition.

3.3.12. Summary: Non-anticipativeness of linear IOM systems. A discrete- or continuous-time linear IOM system with kernel k is non-anticipating if and only if

$$k(t, \tau) = 0 \qquad \text{for} \qquad t < \tau$$

for all t and τ belonging to the time axis on which the system is defined. ∎

3.3.13. Example: Non-anticipativeness of the sliding window averager. The continuous-time sliding window averager of Example 3.2.8(c) is described by the IO relation

$$y(t) = \frac{1}{T_1 + T_2} \int_{t-T_1}^{t+T_2} u(\tau)\, d\tau, \qquad t \in \mathbb{R}.$$

The kernel of this system is

$$k(t, \tau) = \begin{cases} \dfrac{1}{T_1 + T_2} & \text{for } t - T_1 \leq \tau < t + T_2, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \dfrac{1}{T_1 + T_2} & \text{for } \tau - T_2 < t \leq \tau + T_1, \\ 0 & \text{otherwise,} \end{cases} \qquad t, \tau \in \mathbb{R}.$$

A plot of k is given in Fig. 3.15. Inspection shows that $k(t, \tau) = 0$ for $t < \tau$ if and only if $T_2 = 0$, so that by 3.3.12 the averager is non-anticipating if and only if $T_2 = 0$. This agrees with what we concluded in 3.2.8(c).

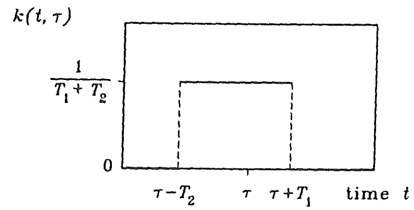


Figure 3.15 Kernel of the sliding window averager. ∎

3.3.14. Remark: Real systems. A *real* IOM system is an IOM system with complex-valued input and output signals, with the property that if the input signal is real, the corresponding output signal is also real. It is easy to see that linear IOM

systems are real if and only if their kernel k is real-valued. The pure delay an sliding window averager are both real systems.

3.3.15. Review: The IO map of sampled discrete-time systems. A *sam* discrete-time system is a discrete-time system whose input and output signals defined on the time axis $\mathbb{Z}(T)$. Sampled IOM systems are linear if and only if t IO map may be represented in the form

$$y(t) = T \sum_{\tau \in \mathbb{Z}(T)} k(t, \tau) u(\tau), \qquad t \in \mathbb{Z}(T),$$

where the function k is again called the *kernel* of the sampled system. The va $k(t, \tau)$ of the kernel k at the point (t, τ) is the response of the system at time t if input is the shifted pulse

$$u(t) = \frac{1}{T} \Delta\left(\frac{t - \tau}{T}\right), \qquad t \in \mathbb{Z}(T).$$

The necessary and sufficient conditions for a linear IOM system to be non-anticip ing or real as stated in 3.3.12 and 3.3.14 also apply to sampled systems.

3.4 CONVOLUTION SYSTEMS

In Section 3.3 it was found that the IO map of any discrete-time linear IOM syste may be represented in the form

$$y(n) = \sum_{m=-\infty}^{\infty} k(n, m) u(m), \qquad n \in \mathbb{Z}, \tag{1}$$

with k the kernel of the system. The value $k(n, m)$ of the kernel k at (n, m) is the response of the system at time n when the input is a unit pulse shifted to time m.

Suppose that besides linear the system is also *time-invariant*. Then by time-invariance for every $d \in \mathbb{Z}$ the response at time $n + d$ to a pulse at time $m + d$ is the same as the response at time n to a pulse at time m, that is,

$$k(n + d, m + d) = k(n, m) \qquad \text{for all} \quad n, m, d \in \mathbb{Z}.$$

Evidently, adding the same number d to each of the arguments of the kernel k does not change its value, which means that k actually is a function of the *difference* of its arguments. Thus, there exists a function h of a *single* variable such that

$$k(n, m) = h(n - m) \qquad \text{for all} \quad n, m \in \mathbb{Z}. \tag{2}$$

This shows that the response at time n of a discrete-time time-invariant system to a pulse at time m is $h(n - m)$ and, hence, only depends on the time *elapsed* since the arrival of the pulse. Substitution of (2) into (1) shows that the response of a discrete-time linear time-invariant IOM system may be expressed as

$$y(n) = \sum_{m=-\infty}^{\infty} h(n - m)u(m), \qquad n \in \mathbb{Z}. \tag{3}$$

This expression defines the output y as the result of an operation on the discrete-time signals h and u that is called (discrete-time) *convolution* and written as

$$y = h * u. \tag{4}$$

We next consider continuous-time systems. It is easy to see that also the kernel k of a continuous-time linear IOM system that is time-invariant is a function

$$k(t, \tau) = h(t - \tau) \qquad \text{for all} \quad t, \tau \in \mathbb{R},$$

of the difference of its arguments only. The response of the system may thus be expressed as

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)u(\tau)\, d\tau, \qquad t \in \mathbb{R}. \tag{5}$$

This operation on the signals h and u is similar to that for the discrete-time case and is called (continuous-time) *convolution*. Again we write

$$y = h * u. \tag{6}$$

For an overview of different types of convolution we refer to Table 3.2 in Section 3.8. The convolution operation is discussed in detail in Section 3.5.

IO maps of the form (4) and (6) are called *convolution maps*, and the corresponding IOM systems are *convolution systems*.

We have shown that every linear time-invariant system is a convolution system. Conversely, every convolution system is linear and time-invariant. We summarize these facts as follows.

3.4.1. Summary: Convolution systems.

A discrete-time linear IOM system with time axis $\mathbb{Z}$ and real- or complex-valued input and output is time-invariant if and only if it is a convolution system.

A continuous-time linear IOM system with time axis $\mathbb{R}$ and real- or complex-valued input and output is time-invariant if and only if it is a convolution system. ∎

Impulse Response

The function h occurring in the convolution maps (3) and (5) is called the *impulse response* of the discrete- or continuous-time system. The reason for this name is that the value $h(t)$ of the function h at time t is the response of the system at time t to a unit pulse (in the discrete-time case) or a delta function (in the continuous-time case) at time 0. It follows that for linear time-invariant systems it is sufficient to know the response h of the system to a pulse or delta function at time 0 to determine the response to *any* input.

Note that we use the name *impulse response* for the function h that describes the response of both discrete-time and continuous-time convolution systems.

Figure 3.16 shows diagrammatically how the output of a convolution system follows by convolution. It also demonstrates how, in particular, the impulse response h is the response to the unit pulse Δ or the delta function δ.

$$u \xrightarrow[\text{with } h]{\text{convolution}} y$$

$$\Delta \text{ or } \delta \xrightarrow[\text{with } h]{\text{convolution}} h$$

Figure 3.16 The output of a convolution system follows by convolving the input with the impulse response h. Top: general input. Bottom: unit input Δ or δ.

Examples

Convolution systems form a major topic in this text. Before starting a detailed discussion of the convolution operation, we give several examples of convolution systems and their impulse responses.

3.4.2. Example: Convolution systems.

(a) *The exponential smoother*. The exponential smoother of Example 1.2.6 is described by the difference equation

$$y(n + 1) = ay(n) + (1 - a)u(n + 1), \qquad n \in \mathbb{Z},$$

where we take the time axis as $\mathbb{Z}$. By repeated substitution it easily follows that if $y(n_o)$ is known for some initial time n_o, the output of the system is given by

$$y(n) = a^{n-n_o}y(n_o) + (1 - a) \sum_{m=n_o+1}^{n} a^{n-m}\, u(m), \qquad n \geq n_o, \qquad n \in \mathbb{Z}.$$

Suppose that the initial condition is $y(n_o) = 0$, and let the initial time n_o approach $-\infty$. Then the response of the system takes the form

$$y(n) = (1-a)\sum_{m=-\infty}^{n} a^{n-m}\,u(m), \qquad n \in \mathbb{Z},$$

assuming that the input u is such that the infinite sum converges. Defining the function h by

$$h(n) = \begin{cases} 0 & \text{for } n < 0, \\ (1-a)a^n & \text{for } n \geq 0, \end{cases}$$
$$= (1-a)a^n \mathbb{1}(n), \qquad n \in \mathbb{Z},$$

we see that on the infinite time axis $\mathbb{Z}$ the system may be represented as the *convolution system* with IO map

$$y(n) = \sum_{m=-\infty}^{\infty} h(n-m)u(m), \qquad n \in \mathbb{Z}.$$

A plot of the impulse response h of the system is given in Fig. 3.17.

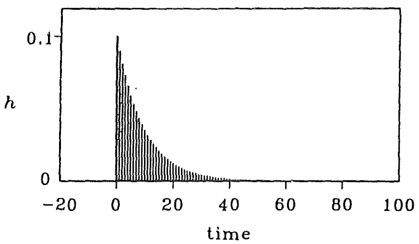


Figure 3.17 The impulse response of the exponential smoother for $a = 0.9$.

(b) *The RC network.* In Example 1.2.7 we found that the solution of the differential equation

$$\frac{dy}{dt}(t) + \frac{1}{RC}y(t) = \frac{1}{RC}u(t), \qquad t \in \mathbb{R},$$

which describes the RC network, for given initial condition $y(t_o)$ at time t_o may be written as

$$y(t) = e^{-(t-t_o)/RC}y(t_o) + \frac{1}{RC}\int_{t_o}^{t} e^{-(t-\tau)/RC}\,u(\tau)\,d\tau, \qquad t \geq t_o.$$

Keeping the initial condition $y(t_o)$ fixed and letting the initial time t_o go to $-\infty$ we see that the output is given by

$$y(t) = \frac{1}{RC}\int_{-\infty}^{t} e^{-(t-\tau)/RC}\,u(\tau)\,d\tau, \qquad t \in \mathbb{R},$$

where we assume that the input u is such that the integral converges. Defining function

$$h(t) = \begin{cases} 0 & \text{for } t < 0, \\ \dfrac{1}{RC}e^{-t/RC} & \text{for } t \geq 0, \end{cases}$$
$$= \frac{1}{RC}e^{-t/RC}\,\mathbb{1}(t), \qquad t \in \mathbb{R},$$

we may represent the system as the continuous-time *convolution system* with IO m

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\,d\tau, \qquad t \in \mathbb{R}.$$

The impulse response h of the system is plotted in Fig. 3.18.

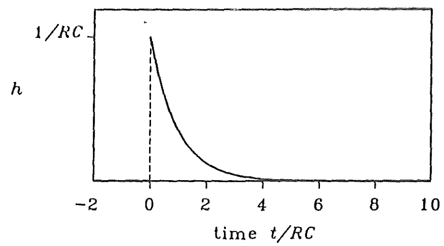


Figure 3.18 The impulse response of the RC network.

(c) *Continuous-time pure delay.* The IO map of the pure delay of Example 3.2.8(a), which is given by

$$y(t) = u(t-\theta), \qquad t \in \mathbb{R},$$

may be rewritten in the form

$$y(t) = \int_{-\infty}^{\infty} \delta(t-\theta-\tau)u(\tau)\,d\tau, \qquad t \in \mathbb{R}.$$

This shows that the delay is a convolution system with the *singular* impulse response

$$h(t) = \delta(t-\theta), \qquad t \in \mathbb{R}. \qquad \blacksquare$$

Non-Anticipativeness of Convolution Systems

Anticipativeness or non-anticipativeness of convolution systems is easily established from the impulse response.

3.4.3. Summary: Non-anticipativeness of convolution systems. A discrete- or continuous-time convolution system with impulse response h is non-anticipating if and only if

$$h(t) = 0 \qquad \text{for} \quad t < 0,$$

with t ranging over the time axis on which the system is defined. ∎

This result is an immediate consequence of 3.3.12 and the fact that $k(t, \tau) = h(t - \tau)$.

3.4.4. Examples: Non-anticipating and anticipating convolution systems. Inspection of the impulse responses of the systems of Example 3.4.2 shows that the exponential smoother and the RC network both are non-anticipating. The pure delay is non-anticipating if and only if the delay θ is nonnegative. ∎

Step Response

The *step response* of a convolution system is its response s when the input u is the unit step $\mathbb{1}$. The step response is closely related to the impulse response. Substitution of $u = \mathbb{1}$ into the IO map yields for the discrete-time case

$$s(n) = (h * \mathbb{1})(n) = \sum_{m=-\infty}^{\infty} h(n-m)\mathbb{1}(m) = \sum_{m=0}^{\infty} h(n-m), \qquad n \in \mathbb{Z}.$$

By the change of variable $k = n - m$ this assumes the form

$$s(n) = \sum_{k=-\infty}^{n} h(k), \qquad n \in \mathbb{Z}.$$

This shows that the step response is the "running sum" of the impulse response. Conversely, by differencing the step response we find

$$s(n) - s(n-1) = h(n), \qquad n \in \mathbb{Z}.$$

This shows how to retrieve the impulse response from the step response.

For the continuous-time case it is easy to obtain the corresponding results.

3.4.5. Summary: Step response of convolution systems.

The step response s of a discrete-time convolution system is the response of the system to the unit step $u = \mathbb{1}$. It is related to the impulse response h by

$$s(n) = \sum_{m=-\infty}^{n} h(m), \qquad h(n) = s(n) - s(n-1), \qquad n \in \mathbb{Z}.$$

The step response s of a continuous-time convolution system is the response of the system to the unit step $u = \mathbb{1}$. It is related to the impulse response h by

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau, \qquad h(t) = \frac{ds(t)}{dt}, \qquad t \in \mathbb{R}.$$ ∎

The step response is sometimes easier to determine than the impulse response. The impulse response may then be obtained from the step response by differencing (in the discrete-time case) or differentiating (in the continuous-time case).

3.4.6. Example: Step responses.

(a) *Exponential smoother.* Given the impulse response of the exponential smoother as found in Example 3.4.2(a), we may determine the step response of the smoother as

$$s(n) = \sum_{m=-\infty}^{n} h(m) = (1-a)\sum_{m=0}^{n} a^m = \begin{cases} 0 & \text{for } n < 0, \\ 1 - a^{n+1} & \text{for } n \ge 0, \end{cases} \qquad n \in \mathbb{Z}.$$

A plot of the step response s is given in Fig. 3.19.

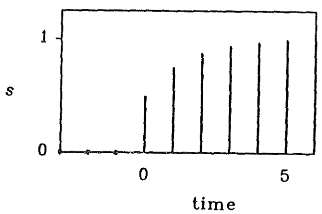


Figure 3.19 Step response of the exponential smoother for $a = 1/2$.

(b) *Pure delay.* The IO map of the continuous-time pure delay of Example 3.2.8(a) is characterized by the relation

$$y(t) = u(t - \theta), \qquad t \in \mathbb{R}.$$

It follows that the step response of the system is

$$s(t) = \mathbb{1}(t - \theta), \qquad t \in \mathbb{R},$$

as sketched in Fig. 3.20. By 3.4.5, the impulse response h of the system follows by differentiation of s and, hence, is given by

$$h(t) = \frac{d}{dt}\mathbb{1}(t - \theta) = \delta(t - \theta), \qquad t \in \mathbb{R}.$$

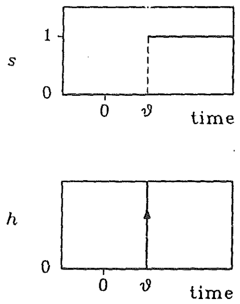


Figure 3.20 Pure delay system. Top: step response. Bottom: impulse response.

This confirms what we found in Example 3.4.2(c). ∎

In conclusion, we briefly review the results of this section for sampled systems.

3.4.7. Review: Sampled convolution systems. The IO map of a linear time-invariant sampled system (see 3.3.15), defined on the time axis $\mathbb{Z}(T)$, may be written as

$$y(t) = T \sum_{\tau \in \mathbb{Z}(T)} h(t - \tau)u(\tau), \qquad t \in \mathbb{Z}(T), \tag{7}$$

with h the *impulse response* of the sampled system. The impulse response is related to the kernel k of the system as $h(t - \tau) = k(t, \tau)$, for t and τ belonging to $\mathbb{Z}(T)$. The expression (7) defines the output y as the sampled convolution

$$y = h * u$$

of the impulse response h and the input u. The impulse response h is the response to the unit pulse

$$u(t) = \frac{1}{T}\Delta\left(\frac{t}{T}\right), \qquad t \in \mathbb{Z}(T).$$

A sampled convolution system is non-anticipating if and only if its impulse response satisfies $h(t) = 0$ for $t < 0$ and $t \in \mathbb{Z}(T)$. The step response s of a sampled convolution system is its response to a unit step $u = \mathbb{1}$. The relation of the step response s to the impulse response h is given by

$$s(t) = T \sum_{\tau \le t, \tau \in \mathbb{Z}(T)} h(\tau), \qquad t \in \mathbb{Z}(T),$$

$$h(t) = \frac{s(t) - s(t - T)}{T}, \qquad t \in \mathbb{Z}(T).$$

Suppose that the sampled convolution system given by (7) is described in *sequen* form by defining the time sequences u^* and y^* as

$$u^*(n) = u(nT), \qquad y^*(n) = y(nT), \qquad n \in \mathbb{Z}(T).$$

Then (7) may be rewritten as

$$y^*(n) = \sum_{m=-\infty}^{\infty} h^*(n - m)u^*(m), \qquad n \in \mathbb{Z},$$

where h^* is given by

$$h^*(n) = T \cdot h(nT), \qquad n \in \mathbb{Z}.$$

This defines a discrete-time convolution system in sequence form. We refer to the impulse response h^* of this system as the *pulse response* of the system (7). For discrete-time convolution systems that are originally described in sequence form the impulse response h and pulse response h^* are identical. ∎

3.5 CONVOLUTION

In this section we present a detailed discussion of the *convolution* operation. Convolution is a binary operation among two discrete- or continuous-time signals, which results in another signal. If x and y are real- or complex-valued time *sequences* defined on the infinite time axis $\mathbb{Z}$, the convolution $z = x * y$ of x and y is the signal given by

$$z(n) = (x * y)(n) = \sum_{m=-\infty}^{\infty} x(n - m)y(m), \qquad n \in \mathbb{Z}, \tag{1}$$

provided the infinite sum exists for all $n \in \mathbb{Z}$. If x and y are two real- or complex-valued *continuous-time* signals defined on the infinite time-axis $\mathbb{R}$, their convolution $z = x * y$ is defined as

$$z(t) = (x * y)(t) = \int_{-\infty}^{\infty} x(t - \tau)y(\tau)d\tau, \qquad t \in \mathbb{R}, \tag{2}$$

provided the integral exists for all $t \in \mathbb{R}$. With this notation, the IO map of a discrete- or continuous-time convolution system may be represented as

$$y = h * u,$$

which explains the name convolution system.

Convolution is *not* a pointwise operation, so that the value of $x * y$ at any given time depends on the *entire* time behavior of the signals x and y. For a good understanding of the convolution it is worthwhile to look at the details of the operation. We consider continuous-time convolution but the same arguments apply to discrete-time convolution. From $z = x * y$ and the definition

$$z(t) = (x * y)(t) = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)d\tau, \qquad t \in \mathbb{R},$$

we see that computing the convolution $z(t)$ at some fixed time $t \in \mathbb{R}$ involves the following steps:

(a) Time reverse the signal x and shift the result forward by t.

(b) Multiply y pointwise by the shifted time reversed signal x and integrate the result to obtain $z(t)$.

These steps need be repeated for every value of t. The procedure shows that the convolution $z = x * y$ may be seen as a *local averaging operation* on y with weights obtained by time reversing and shifting x.

We illustrate the procedure with two examples.

3.5.1. Examples: Convolution.

(a) *Discrete-time convolution*. In Example 3.4.2(a) we found that if we choose the initial condition zero and let the initial time n_o approach $-\infty$, the output of the exponential smoother is given by

$$y = h * u,$$

where h is the one-sided exponential signal plotted in Fig. 3.21(a). Suppose that u is the discrete-time unit step, that is, $u = \mathbb{1}$, as depicted in Fig. 3.21(b) "Locally averaging" u with the time reversed and shifted impulse response h, as shown in Fig. 3.21(c), in this case amounts to taking *past* values of the input u only and weighting them exponentially while averaging. Analytical computation results in

$$y(n) = (h * \mathbb{1})(n) = \begin{cases} 0 & \text{for } n < 0, \\ (1-a)\sum_{m=0}^{n} a^{n-m} = 1 - a^{n+1} & \text{for } n \geq 0, \end{cases} \qquad n \in \mathbb{Z},$$

as shown in Fig. 3.21(d). Because of the shape of h, convolution has the effect of smoothing: u undergoes a step change at time 0, while y gradually changes from 0 to the final value 1.

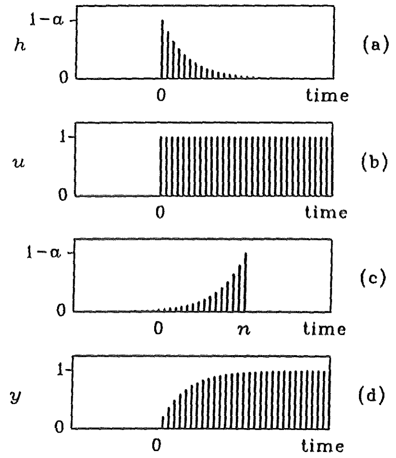


Figure 3.21 Discrete-time convolution of two signals h and u. (a) The signal h. (b) The signal u. (c) The time reversed signal h, shifted by n. (d) the convolution $h * u$.

(b) *Continuous-time convolution*. In Example 3.4.2(b) we found that if the initial time t_o approaches $-\infty$, the output of the RC network is given by the convolution

$$y = h * u,$$

of h and u, with h as shown in Fig. 3.22(a). Since h is a one-sided exponential, convolution of h and u results in exponential weighting of past values of the input. Suppose that u is a rectangular pulse of the form

$$u(t) = \begin{cases} 1 \text{ for } 0 \leq t < a, \\ 0 \text{ otherwise}, \end{cases} \qquad t \in \mathbb{R}.$$

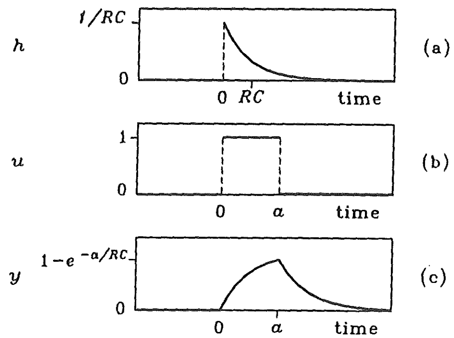


Figure 3.22 Continuous-time convolution of h and u. Top: the signal h. Middle: the signal u. Bottom: the convolution $h * u$.

It follows that

$$y(t) = \frac{1}{RC}\int_{-\infty}^{t} e^{-(t-\tau)/RC}\, u(\tau)\, d\tau$$

$$= \begin{cases} 0 & \text{for } t < 0, \\ 1 - e^{-t/RC} & \text{for } 0 \leq t < a, \\ (1 - e^{-a/RC})e^{-(t-a)/RC} & \text{for } t \geq a. \end{cases}$$

Plots of h, u, and y are given in Fig. 3.22. If one thinks of $h * u$ as the action of h on u, h makes u smoother. The larger RC is, the stronger the effect. ∎

Both in Examples 3.5.1(a) and (b) the effect of convolution is to smooth the input u. Depending on the shape of the impulse response h, also other effects may be achieved. In 3.5.7, for instance, we see that differentiation, which is the opposite of a smoothing operation, may be represented as a convolution.

Properties and Existence of the Convolution

The convolution is a binary operation among time signals. In contrast to all previous binary operations we considered it is not a pointwise operation. Even so, convolution has certain aspects in common with (pointwise) multiplication. In what follows we list the most important properties of the convolution operation.

3.5.2. Summary: Properties of the convolution Let $*$ denote discrete-time or continuous-time convolution, and suppose that x, y, and z are discrete- or continuous-time signals defined on the infinite time axis $\mathbb{Z}$ or $\mathbb{R}$, respectively. Then the following holds.

(a) *Commutativity:* If $x * y$ exists, then

$$x * y = y * x.$$

(b) *Associativity:* If $(x * y) * z$ exists, then

$$(x * y) * z = x * (y * z).$$

(c) *Distributivity:* If $x * y$ and $x * z$ exist, then

$$x * (y + z) = x * y + x * z.$$

(d) *Commutativity of scalar multiplication and convolution:* If $x * y$ exists, then

$$\alpha(x * y) = (\alpha x) * y = x * (\alpha y)$$

for any $\alpha \in \mathbb{C}$.

(e) *Shift property.* Let σ denote the back shift operator. Then, if $x * y$ exists,

$$\sigma^t(x * y) = (\sigma^t x) * y = x * (\sigma^t y)$$

for any $t \in \mathbb{Z}$ in the discrete-time case and any $t \in \mathbb{R}$ in the continuous-ti case.

(f) *Differentiation property.* Let D denote the differentiation operator, that is, if tl continuous-time signal z is differentiable, $Dz(t) = dz(t)/dt$, $t \in \mathbb{R}$. Suppo that the continuous-time convolution $x * y$ exists and is differentiable. Then,

$$D(x * y) = (Dx) * y$$

if x is differentiable and

$$D(x * y) = x * (Dy),$$

if y is differentiable. ∎

The proof of these properties is not difficult.

The caveat in the definition of the convolution (1) or (2) that the infinite sum or integral may not exist is not superfluous. For instance, convolution of two signals x and y that both are constant leads to diverging infinite sums or integrals so that $x * y$ does not exist. We consider some helpful sufficient conditions for the existence of convolutions.

Before stating the existence results we need introduce some terminology. The *support* of a signal x defined on the discrete or continuous time axis $\mathbb{T}$ is the set

$$\{t \in \mathbb{T} \,|\, x(t) \neq 0\}.$$

Thus, the support is the set of time instants on which the system is nonzero. (Actually, the support is the *closure* of this set but we do not elaborate on this.) A signal has *bounded support* if its support is contained in some finite interval. A signal is *right one-sided* if its support is contained in a right semi-infinite interval and *left one-sided* if its support is contained in a left semi-infinite interval. Figure 3.23 gives plots of a signal with bounded support and a right one-sided signal.

A continuous-time signal x is called *locally integrable* if

$$\int_a^b |x(t)|\, dt$$

exists and is finite for every finite a and b.

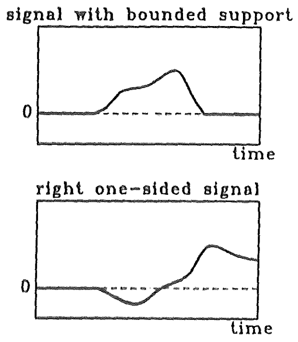


Figure 3.23 Top: a signal with bounded support. Bottom: a right one-sided signal.

3.5.3. Summary: Sufficient conditions for the existence of the convolution. In what follows, x and y are either both time sequences with time axis $\mathbb{Z}$ or both locally integrable continuous-time signals with time axis $\mathbb{R}$.

(a) If x *or* y has bounded support, then $x * y$ exists. If *both* x and y have bounded support, then also $x * y$ has bounded support.
(b) If x *and* y are one-sided (both right or both left), then $x * y$ exists and is also one-sided (in the same direction as x and y).
(c) If $\|x\|_2$ and $\|y\|_2$ are both finite, then $x * y$ exists, and $\|x * y\|_\infty$ is finite (but $\|x * y\|_2$ is not necessarily finite!).
(d) If $\|x\|_1$ and $\|y\|_\infty$ exist, then $x * y$ exists and $\|x * y\|_\infty$ is finite. ∎

3.5.4. Exercise: Proof of 3.5.3. Prove 3.5.3. *Hints:* The proofs of (a), (b), and (d) are straightforward. To prove (c), use the Cauchy-Schwartz inequality. ∎

The statements in parts (a) and (b) concerning the support of the convolution may be made much more specific. The following result makes it possible to delimit the support of a convolution, which is quite useful when doing actual calculations.

3.5.5. Summary: Support of the convolution. Let x and y either be both time-sequences or both continuous-time signals. Suppose that the support of x is contained in some interval $[a, b]$ and that of y in $[c, d]$, with a and c possibly $-\infty$ and b and d possibly ∞. Then the support of $x * y$ is contained in $[a + c, b + d]$. ∎

The proof is left to the reader. Note that if x and y both have bounded support, it immediately follows from 3.5.5 that also $x * y$ has bounded support, as stated in 3.5.3(a). Moreover, if x and y are both one-sided in a certain direction, so is $x * y$, as claimed in 3.5.3(b).

3.5.6. Example: Support of convolutions. In Examples 3.5.1(a) and (b) one of the signals that are convolved always has its support in $[0, \infty)$, while the other either has that same interval as its support or a finite interval of the form $[0, a]$. It follows from 3.5.5 that in both cases the convolution of the two signals has its support in $[0, \infty)$, which agrees with the results that were found. ∎

Convolution With the Unit Functions Δ and δ

Convolving a discrete-time signal x with the unit pulse Δ leaves x unchanged, because

$$\sum_{m=-\infty}^{\infty} x(n - m)\Delta(m) = x(n), \qquad n \in \mathbb{Z},$$

so that $x * \Delta = x$.

Similarly, in the continuous-time case convolving a signal x with the δ-function leaves x unchanged, because

$$\int_{-\infty}^{\infty} x(t - \tau)\delta(\tau)\, d\tau = x(t), \qquad t \in \mathbb{R},$$

so that $x * \delta = x$. In particular,

$$\delta * \delta = \delta.$$

These facts are summarized by the statement that the unit pulse Δ is the *unit* of the discrete-time convolution and the delta function δ that of the continuous-time convolution.

Convolution With Derivatives of the Delta Function

We continue by briefly discussing convolutions with *derivatives* of the delta function. The full development of convolutions of generalized functions is given in Supplement C. Suppose that f is an n times continuously differentiable regular function, with $n \in \mathbb{Z}_+$. Then we have by Property (7) of Table 2.1

$$\begin{aligned}(f * \delta^{(n)})(t) &= \int_{-\infty}^{\infty} f(t - \tau)\delta^{(n)}(\tau)\, d\tau = (-1)^n \frac{d^n}{d\tau^n} f(t - \tau)\Big|_{\tau=0} \\ &= f^{(n)}(t), \qquad t \in \mathbb{R},\end{aligned}$$

with $f^{(n)}$ denoting the nth derivative of f. It follows that

$$f * \delta^{(n)} = f^{(n)}. \tag{3}$$

If f is not an n times differentiable regular function, but *any* generalized function, we still take the convolution of f and $\delta^{(n)}$ to be given by (3). In particular, by letting $f = \delta^{(m)}$, with $m \in \mathbb{Z}_+$, we obtain

$$\delta^{(m)} * \delta^{(n)} = \delta^{(n+m)}, \qquad n, m \in \mathbb{Z}_+.$$

Table 3.1 shows the results of this discussion. The generalized convolution possesses *all* the properties of the continuous-time convolution listed in 3.5.2, including commutavity and the shift and differentiation properties.

TABLE 3.1 CONVOLUTIONS WITH δ-FUNCTIONS

Property	Conditions
(1) $f * \delta^{(n)} = f^{(n)}$	f any regular or generalized function, $n \in \mathbb{Z}_+$
(2) $\delta^{(n)} * \delta^{(m)} = \delta^{(n+m)}$	$n, m \in \mathbb{Z}_+$

3.5.7. Example: The differentiator. The *differentiator* is the continuous-time IOM system with IO relation $y = Du$, with D the differentiation operator. In full,

$$y(t) = \frac{du(t)}{dt}, \qquad t \in \mathbb{R}.$$

It is easily verified that the system is linear and time-invariant and, hence, is a convolution system. To find its impulse response, choose the input as $u = \delta$. The response y is the impulse response h, so that

$$h = D\delta = \delta^{(1)}.$$

By property (1) of Table 3.1 the response s of the system to a unit step may be found as

$$\begin{aligned} s &= h * \mathbb{1} = \delta^{(1)} * \mathbb{1} = D\mathbb{1} \\ &= \delta. \end{aligned}$$

The same result is of course obtained by direct use of the IO relationship

$$s(t) = \frac{d}{dt}\mathbb{1}(t) = \delta(t), \qquad t \in \mathbb{R}.$$ ■

We conclude this section with a brief review of the sampled convolution.

3.5.8. Review: Sampled convolution. The convolution $z = x * y$ of tw sampled signals x and y on the time axis $\mathbb{Z}(T)$ is defined as

$$z(t) = T \sum_{\tau \in \mathbb{Z}(T)} x(t - \tau) y(\tau), \qquad t \in \mathbb{Z}(T).$$

The sampled convolution has all the properties 3.5.2(a)–(e) of the discrete-tim convolution, exists under the same conditions 3.5.3 as the discrete-time convolu tion, and its support may be delimited as in 3.5.5. The unit of the sampled convolu tion is the unit pulse $\Delta(t/T)/T$, $t \in \mathbb{Z}(T)$. ■

3.6 STABILITY OF CONVOLUTION SYSTEMS

Stability is an important subject in system theory. It is also a complex subject and we deal only with some of the many forms and definitions of stability in this book. Roughly, a system is *stable* if *"small"* or *bounded* inputs and initial conditions result in bounded outputs. If a small or bounded input or initial condition produces a response that grows indefinitely the system is *unstable*.

We develop the subject of stability step by step over the next few chapters. At this point we introduce the notion of *bounded-input bounded-output* (BIBO) *stability* of convolution systems.

3.6.1. Definition: BIBO stability of convolution systems. A discrete- or continuous-time convolution system is *bounded-input bounded-output* (BIBO) *stable* if the response y to every input u with finite amplitude has finite amplitude, that is, if $\|u\|_\infty < \infty$ implies $\|y\|_\infty < \infty$. ■

Thus, a convolution system that is BIBO stable is well-behaved in the sense that if the input is bounded, so is the output. Because the response of a convolution system is fully determined by its impulse response h, it is not surprising that the BIBO stability of a convolution system may be verified from its impulse response.

3.6.2. Summary: BIBO stability of convolution systems.

(a) A discrete- or continuous-time convolution system with impulse response h is BIBO stable if and only if the impulse response has finite action, that is, $\|h\|_1 < \infty$.

(b) If the convolution system is BIBO stable, then

$$\|y\|_\infty \le \|h\|_1 \cdot \|u\|_\infty$$

for every input u with finite amplitude, where y is the corresponding output. Equality may always be achieved, for instance by the input given by

$$u(t) = \begin{cases} \overline{h(-t)}/|h(-t)| & \text{if } h(-t) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{T},$$

with $\mathbb{T}$ the time axis of the system.

3.6.3. Proof. We only give the proof for the discrete-time case. That for the continuous-time case is analogous. The output y of the system is given by

$$y(n) = \sum_{m=-\infty}^{\infty} h(n-m)u(m), \qquad n \in \mathbb{Z}.$$

By the triangle inequality for complex numbers

$$|y(n)| = \left| \sum_{m=-\infty}^{\infty} h(n-m)u(m) \right| \leq \sum_{m=-\infty}^{\infty} |h(n-m)u(m)|$$

$$\leq \left(\sum_{m=-\infty}^{\infty} |h(n-m)| \right) \|u\|_\infty, \qquad n \in \mathbb{Z},$$

where we use the fact that $|u(m)| \leq \sup_m |u(m)| = \|u\|_\infty$ for all $m \in \mathbb{Z}$. Substituting $n - m = k$ it follows that

$$|y(n)| \leq \left(\sum_{k=-\infty}^{\infty} |h(k)| \right) \|u\|_\infty = \|h\|_1 \cdot \|u\|_\infty, \qquad n \in \mathbb{Z}.$$

As a result,

$$\|y\|_\infty \leq \|h\|_1 \cdot \|u\|_\infty.$$

Thus, if the action $\|h\|_1$ of the impulse response is finite, the amplitude $\|y\|_\infty$ of the output corresponding to any input u with finite amplitude $\|u\|_\infty$ is bounded from above by $\|h\|_1 \cdot \|u\|_\infty$ and, hence, is finite. This shows that a *sufficient* condition for BIBO stability is that the impulse response h has finite action, and also proves the inequality in (b). Note that it also proves 3.5.3(d).

To prove that $\|h\|_1 < \infty$ is also a *necessary* condition, choose the input as

$$u(m) = \begin{cases} \overline{h(-m)}/|h(-m)| & \text{if } h(-m) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad m \in \mathbb{Z}.$$

If h is real, u reduces to $u(m) = \text{sign}[h(-m)]$, $m \in \mathbb{Z}$, where sign is the sign function introduced in 3.2.6(b). The amplitude of this input is 1 and, hence, the input is certainly bounded. For the corresponding output y we have at time 0

$$y(0) = \sum_{m=-\infty}^{\infty} h(-m)u(m) = \sum_{m=-\infty}^{\infty} |h(-m)| = \|h\|_1.$$

This proves that in (b) equality may be achieved. It also shows that if $\|h\|_1 = \infty$, the output is unbounded and, hence, $\|h\|_1 < \infty$ is not only sufficient but also necessary for BIBO stability. ∎

We illustrate BIBO stability with three examples.

3.6.4. Examples: BIBO stability.

(a) *Exponential smoother*. For the purpose of this example we change the difference equation of the exponential smoother of 1.2.6 to

$$y(n+1) = ay(n) + bu(n+1), \qquad n \in \mathbb{Z},$$

which has the solution

$$y(n) = a^{n-n_o}y(n_o) + b \sum_{m=0}^{n-n_o-1} a^m u(n-m), \qquad n \geq n_o,\ n \in \mathbb{Z}.$$

For $y(n_o) = 0$ and $n_o \to -\infty$ we obtain

$$y = h * u,$$

where the impulse response h is given by

$$h(n) = \begin{cases} 0 & \text{for } n < 0, \\ ba^n & \text{for } n \geq 0, \end{cases} \qquad n \in \mathbb{Z}.$$

Assuming that $b \neq 0$, the action of the impulse response is given by

$$\|h\|_1 = \sum_{k=0}^{\infty} |b| \cdot |a|^k = \begin{cases} \dfrac{|b|}{1-|a|} & \text{for } |a| < 1, \\ \infty & \text{for } |a| \geq 1. \end{cases}$$

It follows from 3.6.2 that the system is BIBO stable if and only if $|a| < 1$.

Indeed, from Example 3.4.6(a) it easily follows that if $a \neq 1$ the step response s of the system is

$$s(n) = \frac{b}{1-a}(1 - a^{n+1})\mathbb{1}(n), \qquad n \in \mathbb{Z}.$$

If $|a| > 1$ the step response increases indefinitely with time, confirming that the system is not BIBO stable for these values of the parameter a.

Exercise. Verify that also for $a = 1$ the step response is unbounded. Determine a bounded input that results in an unbounded output if $a = -1$.

(b) *Integrator*. An *integrator*, when defined on the infinite time axis $\mathbb{R}$, is a continuous-time system whose IO map is given by

$$y(t) = \int_{-\infty}^{t} u(\tau)\, d\tau, \qquad t \in \mathbb{R}.$$

Taking the input u as δ it follows that the impulse response of the system is

$$h(t) = \mathbb{1}(t), \qquad t \in \mathbb{R}.$$

Obviously the action of this system is $\|h\|_1 = \infty$, and, hence, the system is not BIBO stable. Indeed, the step response of the system is

$$s(t) = \int_{-\infty}^{t} \mathbb{1}(\tau)\, d\tau = \text{ramp}(t), \qquad t \in \mathbb{R},$$

which is unbounded.

(c) *Differentiator*. In Example 3.5.7 we saw that the impulse response of the differentiator is

$$h(t) = \delta^{(1)}(t), \qquad t \in \mathbb{R}.$$

The response of the differentiator to the unit step is

$$s(t) = \frac{d}{dt}\mathbb{1}(t) = \delta(t), \qquad t \in \mathbb{R},$$

which is unbounded at time 0. Hence, the differentiator is not BIBO stable.

To see that the impulse response $h = \delta^{(1)}$ of the differentiator has infinite action, one may compute the action of an *approximation* of the derivative of the δ-function. For instance, if we use the triangular approximation $d_n(t) = n\ \text{trian}(nt)$, $t \in \mathbb{R}$, of the δ-function, as proposed in Section 2.5, the action of the derivative $d_n^{(1)}$ of d_n is $2n$, which approaches ∞ as $n \to \infty$. ∎

We conclude with a review of the BIBO stability of sampled convolution systems.

3.6.5. Review: BIBO stability of sampled convolution systems. A sampled convolution system (see 3.4.7) is BIBO stable if and only if the action

$$\|h\|_1 = T \sum_{\tau \in \mathbb{Z}(T)} |h(\tau)|$$

of its impulse response h is finite.

3.7 HARMONIC INPUTS

In this section we discuss the response of convolution systems to *harmonic* inputs. The reason for our interest is that many signals may be *expanded* as a finite or infinite linear combination of harmonic signals, as we show later. If the response to a single harmonic input is known, the linearity of convolution systems may be exploited to find their response to a linear combination of harmonics.

Harmonic Inputs and Frequency Response

As introduced in Chapter 2, the *harmonic* η_f with frequency $f \in \mathbb{R}$ is the complex-valued signal defined by

$$\eta_f(t) = e^{j2\pi f t}, \qquad t \in \mathbb{T},$$

with $\mathbb{T}$ the time axis on which the signal is defined. We consider the continuous-time case, where $\mathbb{T} = \mathbb{R}$. Then the response of a convolution system with impulse response h to the harmonic $u = \eta_f$ is given by

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\, d\tau = \int_{-\infty}^{\infty} h(t-\tau)e^{j2\pi f\tau}\, d\tau, \qquad t \in \mathbb{R},$$

provided the integral exists. By substitution of $t - \tau = \theta$ it follows that

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\theta)e^{j2\pi f(t-\theta)}\, d\theta = \left(\int_{-\infty}^{\infty} h(\theta)e^{-j2\pi f\theta}\, d\theta\right)e^{j2\pi f t} \\ &= \hat{h}(f)e^{j2\pi f t}, \qquad t \in \mathbb{R}. \end{aligned} \tag{1}$$

Here $\hat{h}$ is the function given by

$$\hat{h}(f) = \int_{-\infty}^{\infty} h(\theta)e^{-j2\pi f\theta}\, d\theta, \qquad f \in \mathbb{R}, \tag{2}$$

provided the integral exists. We see from (1) that the response of the convolution system to the harmonic η_f, if it exists, is *again* harmonic and of the form

$$y = \hat{h}(f)\eta_f,$$

where $\hat{h}$ is defined by (2). The function $\hat{h}$ is called the *frequency response function* of the convolution system. Note that $\hat{h}$ usually is a *complex-valued* function.

In the discrete-time case it follows similarly that the response of a convolution system with impulse response h to the harmonic signal $u = \eta_f$ on the time axis $\mathbb{T} = \mathbb{Z}$ also is of the form

$$y = \hat{h}(f)\eta_f,$$

where the frequency response function $\hat{h}$ now is given by

$$\hat{h}(f) = \sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn}, \qquad f \in \mathbb{R}.$$

Figure 3.24 illustrates the results. We summarize as follows.

$$\eta_f \xrightarrow[\text{with } h]{\text{convolution}} \hat{h}(f)\eta_f$$

Figure 3.24 Response of a convolution system to a harmonic input.

3.7.1. Summary: Response of convolution systems to harmonic inputs.

The response of a discrete-time convolution system with time axis $\mathbb{Z}$ and impulse response h to the harmonic input

$$u(n) = e^{j2\pi fn}, \qquad n \in \mathbb{Z},$$

with real frequency f, is

$$y(n) = \hat{h}(f)e^{j2\pi fn}, \qquad n \in \mathbb{Z}.$$

The factor $\hat{h}(f)$ is the value at f of the *frequency response function* $\hat{h}$ of the system, given by

$$\hat{h}(f) = \sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn}, \qquad f \in \mathbb{R},$$

provided it exists.

The response of a continuous-time convolution system with time axis $\mathbb{R}$ and impulse response h to the harmonic input

$$u(t) = e^{j2\pi ft}, \qquad t \in \mathbb{R},$$

with real frequency f, is

$$y(t) = \hat{h}(f)e^{j2\pi ft}, \qquad t \in \mathbb{R}.$$

The factor $\hat{h}(f)$ is the value at f of the *frequency response function* $\hat{h}$ of the system, given by

$$\hat{h}(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt, \qquad f \in \mathbb{R},$$

provided it exists. ■

What follows gives a sufficient condition for the existence of the frequency response function.

3.7.2. Summary: Existence of frequency response functions. The frequency response function $\hat{h}$ of a discrete- or continuous-time convolution system with impulse response h exists and is bounded if the impulse response has finite action, that is, $\|h\|_1 < \infty$. ■

3.7.3. Proof. We give the proof for the discrete-time case only; that for the continuous-time case is similar. Suppose that $\|h\|_1$ is finite. Then

$$|\hat{h}(f)| = \left|\sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn}\right| \le \sum_{n=-\infty}^{\infty} |h(n)|\cdot|e^{-j2\pi fn}| = \sum_{n=-\infty}^{\infty} |h(n)| = \|h\|_1 < \infty,$$

which shows that $\hat{h}(f)$ exists for every $f \in \mathbb{R}$, and moreover is bounded. ■

Note that the condition for the existence of the frequency response function is the same as the necessary and sufficient condition of 3.6.2 for the BIBO stability of the convolution system. Hence, if the system is BIBO stable, its frequency response function exists.

We saw that in the discrete-time case the frequency response function is defined by

$$\hat{h}(f) = \sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn}, \qquad f \in \mathbb{R}.$$

Because the complex exponential $e^{-j2\pi fn}$, with n an integer, is periodic in f with period 1, so is the frequency response function $\hat{h}$. Hence, the response to a harmonic input with frequency f is the same as the response to a harmonic input with frequency $f + m$, with m any integer. This is, of course, an immediate consequence of the *aliasing* effect of 2.2.10, which implies that on the time axis $\mathbb{Z}$ the harmonics η_f and η_{f+m} are indistinguishable for any integer m.

3.7.4. Summary: Periodicity of discrete-time frequency response functions. The frequency response function of a discrete-time convolution system defined on the time axis $\mathbb{Z}$ is periodic with period 1. ■

Because of the periodicity of the frequency response function of a discrete-time convolution system, it is only necessary to specify or display it on a single period, such as the interval $[-½, ½)$. Frequency response functions of continuous-time systems generally are *not* periodic.

We consider a few examples of frequency response functions.

3.7.5. Examples: Frequency response functions.

(a) *Exponential smoother*. From Example 3.4.2(a) the impulse response of the exponential smoother is

$$h(n) = (1 - a)a^n\mathbb{1}(n), \qquad n \in \mathbb{Z}.$$

If $|a| < 1$, the impulse response has finite action, so that by 3.7.2 the system has a frequency response function. It follows that

$$\hat{h}(f) = \sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn} = (1 - a)\sum_{n=0}^{\infty} a^n e^{-j2\pi fn} = (1 - a)\sum_{n=0}^{\infty} (ae^{-j2\pi f})^n$$

$$= \frac{1 - a}{1 - ae^{-j2\pi f}}, \qquad f \in \mathbb{R}.$$

The assumption $|a| < 1$ guarantees the convergence of the infinite sum. Plots of the magnitude and phase of $\hat{h}$ are given in Fig. 3.25 for $a = 1/2$. The periodicity with period 1 is evident.

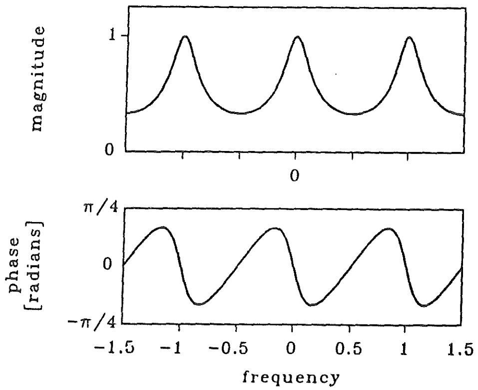


Figure 3.25 Frequency response function of the exponential smoother. Top: magnitude. Bottom: phase.

(b) *RC network.* Next consider the RC network. In Example 3.4.2(b) we saw how the RC network may be considered as a continuous-time convolution system with impulse response

$$h(t) = \frac{1}{RC} e^{-t/RC} \mathbb{1}(t), \qquad t \in \mathbb{R}.$$

The impulse response function has finite action for all $RC > 0$. The frequency response function of the system is

$$\begin{aligned} \hat{h}(f) &= \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t}\, dt = \frac{1}{RC} \int_0^{\infty} e^{-t/RC} e^{-j2\pi f t}\, dt \\ &= \frac{1}{RC} \int_0^{\infty} e^{-(j2\pi f + 1/RC)t}\, dt = -\frac{\dfrac{1}{RC}}{\dfrac{1}{RC} + j2\pi f} e^{-(\frac{1}{RC} + j2\pi f)t} \Big|_0^{\infty} \\ &= \frac{1}{1 + RCj2\pi f}, \qquad f \in \mathbb{R}. \end{aligned}$$

Plots of the magnitude and phase of $\hat{h}$ are shown in Fig. 3.26.

(c) *Integrator.* As we have seen in Example 3.6.4(b), the impulse response of the integrator is

$$h(t) = \mathbb{1}(t), \qquad t \in \mathbb{R}.$$

The unit step has infinite action, so that the condition of 3.7.2 for the existence of the frequency response function $\hat{h}$ is not satisfied. Indeed, the integral

$$\int_0^{\infty} e^{-j2\pi f t}\, dt$$

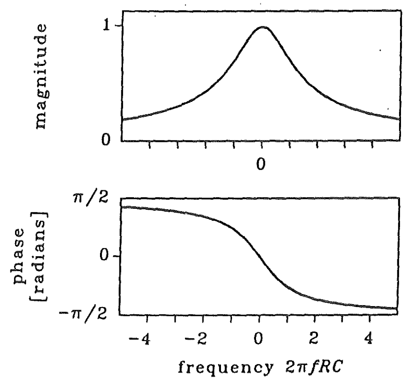


Figure 3.26 Frequency response function of the RC network. Top: magnitude. Bottom: phase.

diverges for every real f, so that the integrator does *not* have a frequency response function in the regular sense. Later (in 7.5.5(a)) we see that the frequency response does exist, but only in the sense of generalized functions.

(d) *Differentiator.* In Example 3.5.7 the impulse response of the differentiator was found to be

$$h(t) = \delta^{(1)}(t), \qquad t \in \mathbb{R}.$$

Although the impulse response has infinite action, it still has a frequency response function, given by

$$\begin{aligned} \hat{h}(f) &= \int_{-\infty}^{\infty} \delta^{(1)}(t) e^{-j2\pi f t}\, dt = j2\pi f \cdot e^{-j2\pi f t}\big|_{t=0} \\ &= j2\pi f, \qquad f \in \mathbb{R}. \end{aligned}$$

This result may be obtained more easily by substituting $u(t) = e^{j2\pi f t}$, $t \in \mathbb{R}$, into the IO relationship $y(t) = du(t)/dt$ of the system. This results in

$$y(t) = j2\pi f \cdot e^{j2\pi f t}, \qquad t \in \mathbb{R},$$

confirming that the frequency response function of the system is $\hat{h}(f) = j2\pi f$, $f \in \mathbb{R}$.

The magnitude and argument of $\hat{h}$ are given by

$$|\hat{h}(f)| = 2\pi |f|, \qquad f \in \mathbb{R},$$

$$\arg(\hat{h}(f)) = \begin{cases} -\pi/2 & \text{for } f < 0, \\ \pi/2 & \text{for } f \geq 0, \end{cases} \qquad f \in \mathbb{R}.$$

as plotted in Fig. 3.27.

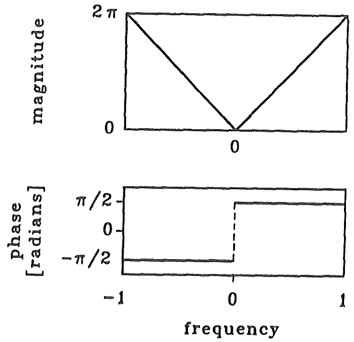


Figure 3.27 Frequency response function of the differentiator. Top: magnitude. Bottom: phase. ∎

Response to Real Harmonic Signals

In the remainder of this section we restrict the discussion to *real* convolution systems. By 3.3.14, real convolution systems are convolution systems whose impulse response h is real-valued. Physical and engineering convolution systems usually are real.

Even if the impulse response is real, the frequency response function generally is a complex-valued function. It has several symmetry properties, enumerated in the following.

3.7.6. Summary: Symmetry properties of the frequency response function of real convolution systems. The frequency response function $\hat{h}$ of a *real* discrete- or continuous-time convolution system is *conjugate symmetric,* that is,

$$\hat{h}(-f) = \overline{\hat{h}(f)} \qquad \text{for all} \quad f \in \mathbb{R},$$

with the overbar denoting the complex conjugate. This implies that

(a) the magnitude $|\hat{h}|$ of $\hat{h}$ is *even,* that is,

$$|\hat{h}(-f)| = |\hat{h}(f)| \qquad \text{for all} \quad f \in \mathbb{R},$$

(b) the phase $\arg(\hat{h})$ of $\hat{h}$ is *odd,* that is,

$$\arg(\hat{h}(-f)) = -\arg(\hat{h}(f)) \qquad \text{for all} \quad f \in \mathbb{R}.$$ ∎

These properties are easy to prove. The plots of Figures 3.25, 3.26, and 3.27 show clearly that the frequency response functions of the exponential smoother, RC network, and differentiator, which are real systems, have even magnitude and odd phase.

Because of the symmetry properties, it is enough to specify the frequency response function of a real system for *nonnegative* frequencies only. For this reason the

negative frequency part is often omitted in plots. Because the frequency response function of *discrete-time* systems according to 3.7.4 is periodic with period 1, it is sufficient to plot the frequency response of discrete-time real convolution systems on the frequency interval $[0, \frac{1}{2}]$ only.

For real discrete- and continuous-time convolution systems we may give an illuminating interpretation of the *magnitude* $|\hat{h}(f)|$ and *phase* $\arg(\hat{h}(f))$ of the frequency response function at some frequency f in terms of the response of the system to a *real* harmonic signal with frequency f. Let x be the real harmonic signal

$$x(t) = \alpha_x \cos(2\pi f t + \phi_x), \qquad t \in \mathbb{T},$$

with amplitude α_x and phase ϕ_x. We recall from Section 2.2 that the *phasor* of this signal is the complex number

$$a_x = \alpha_x e^{j\phi_x},$$

and that the relation between the real harmonic signal x and its phasor a_x is

$$x(t) = \mathrm{Re}[a_x\, e^{j2\pi f t}], \qquad t \in \mathbb{R}.$$

3.7.7. Summary: Response of real convolution systems to real harmonics. The response of a real discrete- or continuous-time convolution system with frequency response function $\hat{h}$ to the real harmonic input

$$u(t) = \alpha_u \cos(2\pi f t + \phi_u) = \mathrm{Re}[a_u\, e^{j2\pi f t}], \qquad t \in \mathbb{T},$$

with $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$, respectively, $\alpha_u \geq 0$ and ϕ_u real numbers and $a_u = \alpha_u e^{j\phi_u}$ the phasor of u, is the real harmonic signal

$$y(t) = \alpha_y \cos(2\pi f t + \phi_y) = \mathrm{Re}[a_y\, e^{j2\pi f t}], \qquad t \in \mathbb{T}.$$

The phasor a_y of y is given by

$$a_y = \hat{h}(f) a_u.$$

It follows that the amplitude α_y and phase ϕ_y of the output are

$$\alpha_y = |\hat{h}(f)| \cdot \alpha_u,$$
$$\phi_y = \phi_u + \arg(\hat{h}(f)).$$

3.7.8. Proof. The proof is not difficult. The idea is to write the real harmonic input u as a sum of complex harmonics and then use linearity to obtain the response of the system. The phasor of the input is $a_u = \alpha_u e^{j\phi_u}$, and, hence, we may write the input as

$$u = \mathrm{Re}(a_u \eta_f) = \tfrac{1}{2}(a_u \eta_f + \overline{a_u \eta_f}) = \tfrac{1}{2}(a_u \eta_f + \overline{a}_u \eta_{-f}).$$

This shows that the input is the sum of two complex harmonic signals, one with frequency f and the other with frequency $-f$. By linearity, the response to this input is

$$y = \tfrac{1}{2}[\hat{h}(f)a_u\eta_f + \hat{h}(-f)\bar{a}_u\eta_{-f}].$$

Because by assumption the system is real, by conjugate symmetry $\hat{h}(-f) = \overline{\hat{h}(f)}$, and, hence,

$$y = \tfrac{1}{2}[\hat{h}(f)a_u\eta_f + \overline{\hat{h}(f)a_u\eta_f}] = \mathrm{Re}[\hat{h}(f)a_u\eta_f].$$

This shows that the output is again real harmonic, with phasor $a_y = \hat{h}(f)a_u$. It follows that the amplitude and phase of the output are given by

$$\alpha_y = |a_y| = |\hat{h}(f)|\cdot|a_u| = |\hat{h}(f)|\cdot\alpha_u,$$

$$\phi_y = \arg(a_y) = \arg(\hat{h}(f)) + \arg(a_u) = \arg(\hat{h}(f)) + \phi_u.$$

This completes the proof. ∎

This result shows the interpretation of the magnitude and phase plots of Figs. 3.25–3.27. The magnitude plots indicate how much the amplitudes of real harmonic signals are amplified or attenuated, depending on their frequency. The phase plots give the corresponding phase shifts.

3.7.9. Example: Response of the RC network to a real harmonic. From Example 3.7.5(b) the frequency response function of the RC network is

$$\hat{h}(f) = \frac{1}{1 + RCj2\pi f}, \qquad f \in \mathbb{R}.$$

The magnitude and phase of $\hat{h}$ are given by

$$|\hat{h}(f)| = \frac{1}{\sqrt{1 + R^2C^24\pi^2f^2}}, \qquad f \in \mathbb{R},$$

$$\arg(\hat{h}(f)) = -\mathrm{atan}(RC2\pi f), \qquad f \in \mathbb{R}.$$

If the input is a real harmonic with frequency $f_o = 1/RC2\pi$, we have $\hat{h}(f_o) = 1/(1 + j)$, and the magnitude and phase are

$$|\hat{h}(f_o)| = \tfrac{1}{2}\sqrt{2}, \qquad \arg(\hat{h}(f_o)) = -\frac{\pi}{4}.$$

Thus, the response to the input

$$u(t) = \cos(2\pi f_o t), \qquad t \in \mathbb{R},$$

is

$$y(t) = \tfrac{1}{2}\sqrt{2}\cos\left(2\pi f_o t - \frac{\pi}{4}\right), \qquad t \in \mathbb{R}.$$

Input and output are shown in Fig. 3.28. The input is attenuated by a factor $\tfrac{1}{2}\sqrt{2}$ 0.707 and delayed by one eighth of a period (because $\pi/4$ is one eighth of 2π).

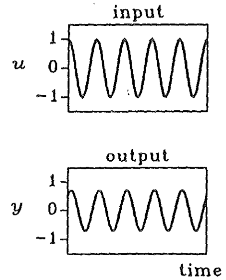


Figure 3.28 Response of the RC network to a real harmonic input. Top: input. Bottom: output. ∎

Engineering Significance of Frequency Response

Convolution systems are often used as *filters*. Filters modify harmonic signals by certain desired frequency response functions. A filter that attenuates high-frequency harmonics relative to low-frequency harmonics is called a *low-pass* filter. Low-pass filters have the general effect of *smoothing* the input. Conversely, if low-frequency harmonics are attenuated compared with high-frequency harmonics, the filter is *high-pass*. High-pass filters remove slowly varying components of the input. A *band-pass* filter passes harmonic signals with frequencies in a certain band and attenuates harmonics with all other frequencies. A *band-stop* filter, finally, rejects harmonics with frequencies in a certain band, and passes all other harmonics. Figure 3.29 illustrates these notions.

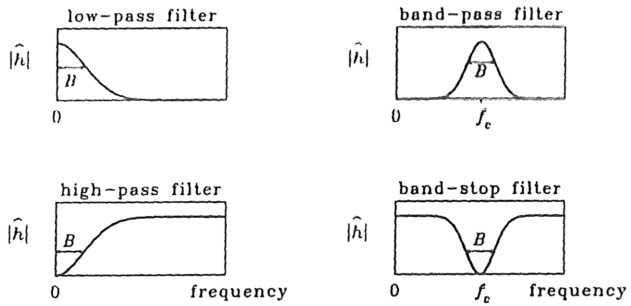


Figure 3.29 Low-pass, high-pass, band-pass, and band-stop filters. B is the bandwidth, f_c the center frequency.

The *bandwidth* B of a low-pass filter specifies the band $[0, B]$ transmitted by the filter. If the filter does not cut off sharply, the bandwidth is not well defined. In such a case it is usually taken as the frequency where the magnitude of the frequency response function has fallen off to a certain fraction, often taken as $\frac{1}{2}\sqrt{2}$—approximately 70%—of the peak value. The bandwidth of a high-pass filter is the width of the band that is suppressed by the filter. To characterize band-pass and band-stop filters it is customary to distinguish the *center frequency* f_c, which more or less specifies the middle of the band, and the bandwidth B. These quantities are shown in Fig. 3.29.

The frequency response functions of Figures 3.25 and 3.26 both represent low-pass filters. For a correct interpretation of the low-, high- or band-pass character of discrete-time filters, their frequency response function should be considered on the frequency interval $[0, \frac{1}{2}]$.

Since the range of the magnitude of frequency response functions usually encompasses several orders of magnitude, the magnitude is often plotted *logarithmically*. In engineering applications it is common to plot the logarithm of the *square* $|\hat{h}|^2$ of the magnitude, because this is a measure for the transfer of the *power* of the input signal. If $\hat{h}$ is physically dimensionless, $\log_{10}(|\hat{h}|^2)$ is expressed in *bel*. Usually this number is converted to *decibel* (dB), which is one tenth of a bel. Thus, the magnitude of $\hat{h}$ is expressed as $10 \log_{10}(|\hat{h}|^2) = 20 \log_{10}(|\hat{h}|)$ dB. The following little table compares how factors of different magnitudes are expressed in dB:

Absolute value	dB
100	40
10	20
1	0
0.1	−20
0.01	−40

Decibels are often also used when $\hat{h}$ is not physically dimensionless. The dB scale then refers to the ratio of $|\hat{h}|^2$ and the relevant physical unit. An advantage of expressing the magnitude of $\hat{h}$ logarithmically is that *multiplication* of the magnitudes of frequency response functions simplifies to *addition*. As seen in Section 3.9, frequency response functions need be multiplied whenever systems are connected in series.

> The bel and decibel are named after the American inventor Alexander Graham Bell (1847–1922.)

Because commonly frequency also ranges over several orders of magnitude, in the continuous-time case frequency is also often plotted logarithmically. The engineering unit corresponding to a logarithmic frequency scale is the *octave*, which measures a factor *two* between two frequencies. Thus, 3 Hz and 6 Hz are one octave apart. The scientific unit is *decade*, which measures a factor *ten*. The frequencies 1.1 Hz and 11 Hz are one decade apart.

We defined the frequency response function $\hat{h}$ as a function of frequency f. Sometimes, however, it is convenient to consider and plot $\hat{h}$ as a function $\hat{h}(f) = \hat{h}(\omega/2\pi)$ of the *angular frequency* $\omega = 2\pi f$.

3.7.10. Example: RC network as low- and high-pass filter.

(a) *Low-pass*. So far we considered the RC network with the voltage v_C across the capacitor as output, as shown in Fig. 3.30. In Example 3.7.5(b) we found that its frequency response function (here denoted as $\hat{h}_C$) is

$$\hat{h}_C(f) = \frac{1}{1 + RCj2\pi f}, \qquad f \in \mathbb{R}.$$

In Fig. 3.26 the magnitude and phase of $\hat{h}$ are plotted as a function of the frequency f with linear scales. The plot shows that the RC network is a low-pass filter.

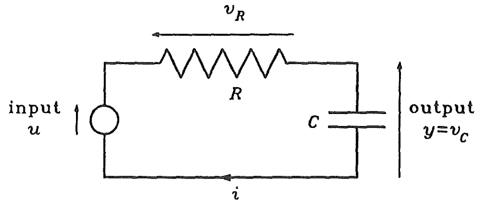


Figure 3.30 RC network.

In Fig. 3.31 the same plot is given with logarithmic scales for frequency and magnitude. It confirms the low-pass nature. The plot is repeated in Fig. 3.32 with the modifications that the frequency response is plotted as a function

$$\hat{h}(\omega/2\pi) = \frac{1}{1 + RCj\omega}, \qquad \omega \in \mathbb{R},$$

of the angular frequency ω on a logarithmic frequency scale, while the magnitude is plotted in dB and the phase in degrees.

The magnitude plot has a low-frequency asymptote 1, corresponding to 0 dB. The high-frequency asymptote of the magnitude is $1/RC\omega$, which corresponds to $-20\log_{10}(\omega) - 20\log_{10}(RC)$ dB, resulting in a straight line with slope -20 dB/decade in the magnitude plot of Fig. 3.32. The low- and high-frequency asymptote intersect at the angular frequency $\omega = 1/RC$, which may be taken as the bandwidth of the low-pass filter.

The low-frequency asymptote of the phase is 0° and the high-frequency asymptote $-90°$.

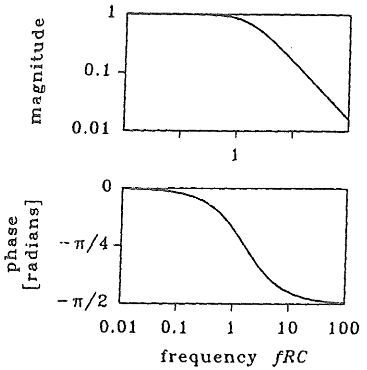


Figure 3.31 Frequency response of the RC network. Top: magnitude. Bottom: phase.

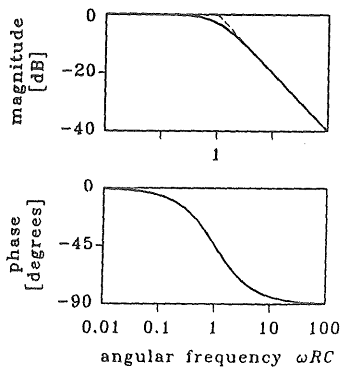


Figure 3.32 Frequency response of the RC network. Top: magnitude. Bottom: phase.

(b) *High-pass*. We also consider the situation that the voltage v_R across the resistor is the output of the network. Since $u = v_R + v_C$, it follows that

$$v_R = u - v_C.$$

If the input is the harmonic η_f we have $u = \eta_f$ and $v_C = \hat{h}_C(f)\eta_f$, so that

$$v_R = [1 - \hat{h}_C(f)]\eta_f.$$

This shows that if the voltage across the resistor is the output of the network, the frequency response function $\hat{h}_R$ is

$$\hat{h}_R(f) = 1 - \hat{h}_C(f) = 1 - \frac{1}{1 + RCj2\pi f} = \frac{RCj2\pi f}{1 + RCj2\pi f}, \qquad f \in \mathbb{R}.$$

The magnitude and phase of this frequency response function are given by

$$|\hat{h}_R(f)| = \frac{RC2\pi|f|}{\sqrt{1 + R^2C^24\pi^2f^2}},$$

$$\arg(\hat{h}_R(f)) = \frac{\pi}{2} - \operatorname{atan}(RC2\pi f), \qquad f \in \mathbb{R}.$$

The plot of the frequency response function of $\hat{h}_R$ given in Fig. 3.33 shows that the system is a *high-pass* filter. Figure 3.34 shows the same plot as a function

$$\hat{h}_R(\omega/2\pi) = \frac{RCj\omega}{1 + RCj\omega}, \qquad \omega \in \mathbb{R},$$

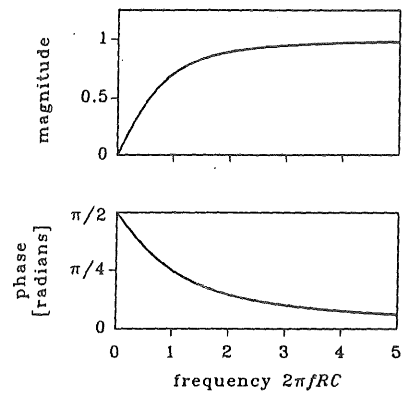


Figure 3.33 Frequency response function of the RC network as a high-pass filter. Top: magnitude. Bottom: phase.

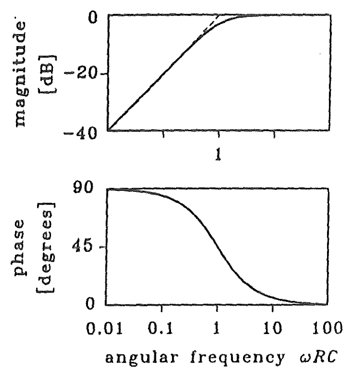


Figure 3.34 Frequency response of the RC network as a high-pass filter. Top: magnitude. Bottom: phase.

of angular frequency, with logarithmic frequency and amplitude scales. The low frequency asymptote of the magnitude is $RC\omega$, corresponding to $20 \log_{10}(\omega) + 20 \log_{10}(RC)$ dB, which is a straight line with slope 20 dB/decade. The high-frequency asymptote is 1, corresponding to 0 dB. The low- and high-frequency asymptotes intersect at the angular frequency $\omega = 1/RC$, which may be taken as the bandwidth. The low frequency asymptote of the phase is 90° and the high-frequency asymptote 0°.

The quality of the low- and high-pass filters represented by the RC network is low, in the sense that the filter does not cut off sharply. To obtain better filters, higher-order networks are needed. The construction of band-pass and band-stop filters also requires higher-order systems. ∎

As usual, we conclude with a review of the results of this section for sampled systems.

3.7.11. Review: Frequency response of sampled convolution systems. The response of a sampled convolution system with time axis $\mathbb{Z}(T)$ to the harmonic input $u(t) = e^{j2\pi ft}$, $t \in \mathbb{Z}(T)$, is given by

$$y(t) = \hat{h}(f)e^{j2\pi ft}, \qquad t \in \mathbb{Z}(T).$$

The frequency response $\hat{h}$ follows from the impulse response h of the system as

$$\hat{h}(f) = T \sum_{t \in \mathbb{Z}(T)} h(t)e^{-j2\pi ft}, \qquad f \in \mathbb{R}.$$

The frequency response function $\hat{h}$ exists if the impulse response h has finite action $\|h\|_1$. The frequency response function is *periodic* with period $1/T$. If the system is *real* (i.e., the impulse response h is real-valued), then the frequency response function possesses the symmetry properties of 3.7.6. The response of the system to real harmonic inputs may be obtained as in 3.7.7. ∎

3.7.12. Examples: Sampled version of the exponential smoother. In Example 3.7.5(a) we considered the exponential smoother on the time axis $\mathbb{Z}$. On the time axis $\mathbb{Z}(T)$ the exponential smoother is described by the difference equation

$$y(t + T) = ay(t) + (1 - a)u(t + T), \qquad t \in \mathbb{Z}(T).$$

For given initial condition $y(t_o)$, with $t_o \in \mathbb{Z}(T)$, the solution of the difference equation is

$$y(t) = a^{\frac{t-t_o}{T}} y(t_o) + (1 - a) \sum_{\substack{t_o<\tau\le t,\\ \tau\in\mathbb{Z}(T)}} a^{\frac{t-\tau}{T}} u(\tau), \qquad t \geq t_o, \qquad t \in \mathbb{Z}(T).$$

Taking $y(t_o) = 0$ and letting $t_o \to -\infty$ we obtain

$$y(t) = (1 - a) \sum_{\tau\le t,\tau\in\mathbb{Z}(T)} a^{\frac{t-\tau}{T}} u(\tau), \qquad t \in \mathbb{Z}(T).$$

This shows that the smoother is a sampled convolution system with impulse response

$$h(t) = \frac{1 - a}{T} a^{\frac{t}{T}} \mathbb{1}(t), \qquad t \in \mathbb{Z}(T).$$

The impulse response has finite action $\|h\|_1$ if and only if $|a| \leq 1$. If $|a| \leq 1$, the frequency response function is given by

$$\begin{aligned}
\hat{h}(f) &= (1 - a) \sum_{t\ge0,t\in\mathbb{Z}(T)} a^{\frac{t}{T}} e^{-j2\pi ft} \\
&= (1 - a) \sum_{k=0}^{\infty} a^k e^{-j2\pi fkT} = (1 - a) \sum_{k=0}^{\infty} (ae^{-j2\pi fT})^k \\
&= \frac{1 - a}{1 - ae^{-j2\pi fT}}, \qquad f \in \mathbb{R}.
\end{aligned}$$

The frequency response function is periodic in the frequency f with period $1/T$. Because the impulse response h is real, $\hat{h}$ is conjugate symmetric, so that it is sufficient to consider the frequency response on the frequency interval $[0, 1/2T]$. A plot of the magnitude and phase of $\hat{h}$ is given in Fig. 3.35 for $a = 1/2$.

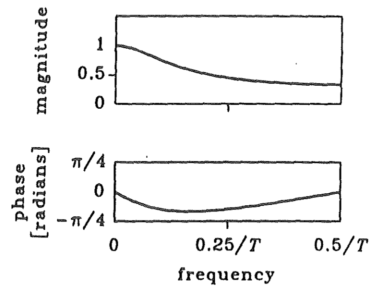


Figure 3.35 Frequency response of the exponential smoother as a sampled convolution system. Top: magnitude. Bottom: phase.

The exponential smoother is a low-pass filter of poor quality. Figure 3.35 shows that for $a = 1/2$ the 70% cut-off frequency (i.e., the frequency at which the magnitude of $\hat{h}$ is 70% of the peak value), is about $0.125/T$. The cut-off frequency thus is proportional to the sampling rate $1/T$. ∎

3.8 PERIODIC INPUTS

In the preceding section we saw that the response of convolution systems to a harmonic input is also harmonic. We now go a step farther, and show that the response of convolution systems to *any* periodic input is periodic. Furthermore, to compute the response to a periodic input, the convolution may be reduced to a special form, called *cyclical* convolution. The cyclical convolution is defined on a *finite* rather than an infinite time axis.

We start with the continuous-time case. Let the continuous-time periodic signal u with period P be the input to a convolution system with impulse response h. Then if the output y exists, we may write

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(t - \tau)u(\tau)\, d\tau \\
&= \cdots + \int_{-P}^{0} h(t - \tau)u(\tau)\, d\tau + \int_{0}^{P} h(t - \tau)u(\tau)\, d\tau \\
&\qquad + \int_{P}^{2P} h(t - \tau)u(\tau)\, d\tau + \cdots, \qquad t \in \mathbb{R}.
\end{aligned}$$

By suitable changes of variable this may be brought into the form

$$\begin{aligned}
y(t) &= \cdots + \int_{0}^{P} h(t - \tau + P)u(\tau - P)\, d\tau + \int_{0}^{P} h(t - \tau)u(\tau)\, d\tau \\
&\qquad + \int_{0}^{P} h(t - \tau - P)u(\tau + P)\, d\tau + \cdots \\
&= \sum_{k=-\infty}^{\infty} \int_{0}^{P} h(t - \tau - kP)u(\tau + kP)\, d\tau, \qquad t \in \mathbb{R}.
\end{aligned}$$

By using the fact that u is periodic with period P, this may be rewritten as

$$\begin{aligned} y(t) &= \int_0^P \left(\sum_{k=-\infty}^{\infty} h(t - \tau - kP) \right) u(\tau)\, d\tau \\ &= \int_0^P h_{\text{per}}(t - \tau) u(\tau)\, d\tau, \qquad t \in \mathbb{R}, \end{aligned} \tag{1}$$

where

$$h_{\text{per}}(t) = \sum_{k=-\infty}^{\infty} h(t - kP), \qquad t \in \mathbb{R}.$$

The signal h_{per} is called the *periodic extension* of h. It is easy to see that h_{per} is periodic with period P. Also y is periodic with period P, because by the periodicity of h_{per} we have from (1)

$$\begin{aligned} y(t + P) &= \int_0^P h_{\text{per}}(t + P - \tau) u(\tau)\, d\tau \\ &= \int_0^P h_{\text{per}}(t - \tau) u(\tau)\, d\tau = y(t), \qquad t \in \mathbb{R}. \end{aligned}$$

In (1) the integration is carried out over a finite interval rather than the entire time axis $\mathbb{R}$. Moreover, because the output y is periodic with period P, it is sufficient to evaluate it over one period, say on the interval $[0, P)$. By the periodicity of h_{per} we may write

$$y(t) = \int_0^P h_{\text{per}}((t - \tau) \bmod P) u(\tau)\, d\tau, \qquad t \in [0, P). \tag{2}$$

Mod is the modulo operator. If a and b are real numbers with b positive, then $a \bmod b = a - kb$, with the integer k such that $0 \le a - kb < b$.

In (2), the three signals y, h_{per} and u that are involved need only be given on the interval $[0, P)$. To emphasize this, let U be a signal defined on the *finite* time axis $[0, P)$ such that

$$U(t) = u(t) \quad \text{for } 0 \le t < P.$$

U is called the *one-period restriction* of the periodic signal u. Similarly, let H be the one-period restriction of h_{per} and Y that of y. Then it follows from (2) that

$$Y(t) = \int_0^P H((t - \tau) \bmod P) U(\tau)\, d\tau, \qquad t \in [0, P).$$

This operation on the finite-time signal H and U is called the (continuous-time *cyclical convolution* of H and U. We write

$$Y = H \odot U.$$

Table 3.2 reviews the different types of convolutions encountered so far. It also includes the sampled cyclical convolution introduced in 3.8.13.

TABLE 3.2 SUMMARY OF CONVOLUTIONS

Type of convolution	Time axis	Convolution: *shorthand*	Convolution: *longhand*
discrete-time cyclical convolution	$\underline{N}$	$x \odot y$	$\sum_{m=0}^{N-1} x((n - m) \bmod N) y(m)$
discrete-time (regular) convolution	$\mathbb{Z}$	$x * y$	$\sum_{m=-\infty}^{\infty} x(n - m) y(m)$
sampled cyclical convolution	$\underline{N}(T)$	$x \odot y$	$T \sum_{\tau \in \underline{N}(T)} x((t - \tau) \bmod NT) y(\tau)$
sampled (regular) convolution	$\mathbb{Z}(T)$	$x * y$	$T \sum_{\tau \in \mathbb{Z}(T)} x(t - \tau) y(\tau)$
continuous-time cyclical convolution	$[0^-, P)$	$x \odot y$	$\int_{0^-}^{P} x((t - \tau) \bmod P) y(\tau)\, d\tau$
continuous-time (regular) convolution	$\mathbb{R}$	$x * y$	$\int_{-\infty}^{\infty} x(t - \tau) y(\tau)\, d\tau$

This introduction shows how the response of continuous-time convolution systems to periodic inputs may be obtained by cyclical convolution. A very similar derivation, with sums replacing integrals, applies to discrete-time systems. Again, one period of the response of the system to a periodic input may be obtained by cyclical convolution of one period of the periodic extension of the impulse response and one period of the input.

The results obtained here are summarized in 3.8.7(a) and 3.8.9. Before arriving there we introduce the periodic extension, one-period restriction, and cyclical convolution more formally.

Periodic Extension and One-Period Restriction

By periodic extension we manufacture a periodic signal from an infinite-time signal. One-period restriction turns one period of a periodic signal into a finite-time signal.

3.8.1. Definitions: Periodic extension and one-period restriction.
(a) *Periodic extension.* Let x be a complex-valued signal on the discrete time axis $\mathbb{T} = \mathbb{Z}$ or the continuous time axis $\mathbb{T} = \mathbb{R}$. Then for given $P \in \mathbb{T}$, with $P > 0$, the *periodic extension with period P* of x is the signal x_{per} defined by

$$x_{\text{per}}(t) = \sum_{k=-\infty}^{\infty} x(t - kP), \qquad t \in \mathbb{T},$$

provided the sum exists for all $t \in \mathbb{T}$.

(b) *One-period restriction.* Let x be a periodic signal with period $P \in \mathbb{T}$ defined on the infinite discrete time axis $\mathbb{T} = \mathbb{Z}$ or the infinite continuous time axis $\mathbb{T} = \mathbb{R}$. Then the *one-period restriction* of x is the signal X defined on the finite time axis $\mathbb{T}_P = [0, P) \cap \mathbb{T}$ given by

$$X(t) = x(t) \qquad \text{for} \quad t \in \mathbb{T}_P. \qquad \blacksquare$$

Periodic extension of a continuous-time signal is illustrated in Fig. 3.36.

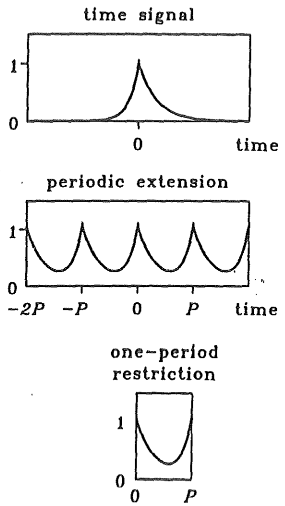


Figure 3.36 Top: an infinite-time signal. Middle: its periodic extension with period P. Bottom: the one-period restriction of the periodic extension.

3.8.2. Exercise: Properties of the periodic extension.
(a) *Existence.* Show that if the infinite-time signal x has finite action its periodic extension exists for any period $P > 0$.

(b) *Periodic extension of a finite-support signal.* Suppose that the support of the infinite-time signal x with time axis $\mathbb{T}$ is contained in the finite interval $[0, P)$. Prove that the periodic extension of x with period P always exists and coincides with x on $[0, P) \cap \mathbb{T}$. $\blacksquare$

Cyclical Convolution

We continue with the definitions of the discrete- and continuous-time cyclical convolutions.

3.8.3. Definition: Cyclical convolution.

Let x and y be two complex-valued discrete-time signals defined on the finite time axis $\underline{N} = \{0, 1, \cdots, N-1\}$. Then, the (discrete-time) *cyclical convolution* $x \odot y$ of x and y is defined on the same time axis and given by

$$(x \odot y)(n) = \sum_{m \in \underline{N}} x((n - m) \bmod N) y(m),$$

for $n \in \underline{N}$.

Let x and y be two complex-valued continuous-time signals defined on the finite time axis $[0, P)$. Then, the (continuous-time) *cyclical convolution* $x \odot y$ of x and y is defined on the same time axis and given by

$$(x \odot y)(t) = \int_0^P x((t - \tau) \bmod P) y(\tau)\, d\tau,$$

for $t \in [0, P)$. $\blacksquare$

The cyclical convolution is quite similar to ordinary convolution. The principal difference is that if the argument of one of the two signals that are convolved cyclically moves out of the signal axis, it is cyclically shifted back into the signal axis.

3.8.4. Example: Cyclical convolution. Consider the signals x and y defined on the time axis $\underline{4} = \{0, 1, 2, 3\}$, given by $x = (0, 1, 2, 3)$ and $y = (4, 5, 6, 7)$. Then, $z := x \odot y$ is defined on the same time axis, with

$$\begin{aligned}
z(0) &= 0\cdot 4 + 3\cdot 5 + 2\cdot 6 + 1\cdot 7 = 34,\\
z(1) &= 1\cdot 4 + 0\cdot 5 + 3\cdot 6 + 2\cdot 7 = 36,\\
z(2) &= 2\cdot 4 + 1\cdot 5 + 0\cdot 6 + 3\cdot 7 = 34,\\
z(3) &= 3\cdot 4 + 2\cdot 5 + 1\cdot 6 + 0\cdot 7 = 28.
\end{aligned} \qquad \blacksquare$$

3.8.5. Exercise: Matrix representation of the discrete-time cyclical convolution. If u is a complex-valued discrete-time signal with time axis $\underline{N}$, denote by $\vec{u}$ the N-dimensional column vector

$$\vec{u} = \text{col}[u(0), u(1), \cdots, u(N-1)]$$

and by C_u the $N \times N$ matrix

$$C_u = \begin{bmatrix} u(0) & u(N-1) & \cdots & u(1) \\ u(1) & u(0) & \cdots & u(2) \\ \cdots & \cdots & \cdots & \cdots \\ u(N-1) & u(N-2) & \cdots & u(0) \end{bmatrix}.$$

Note that each column of C_u is a cyclical shift of the preceding column. A matrix with this structure is called a *circulant matrix*. Prove for any two signals x and y defined on the time axis $\underline{N}$ and $z = x \odot y$ that

$$\vec{z} = C_x\vec{y} = C_y\vec{x}.$$

Suppose for instance that the signals x and y are represented by $\vec{x} = \text{col}(1, 1, 0, 0)$ and $\vec{y} = \text{col}(8/15, 4/15, 2/15, 1/15)$. Then if $z = x \odot y$ we have

$$\vec{z} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 8/15 \\ 4/15 \\ 2/15 \\ 1/15 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \\ 2/5 \\ 1/5 \end{bmatrix}.$$

The properties of the cyclical convolution parallel those of the ordinary convolution.

3.8.6. Summary: Properties of the cyclical convolution. Let $\odot$ denote the cyclical convolution of signals defined on the finite discrete time axis $\underline{N}$ or the finite continuous time axis $[0, P)$.

(a) All the properties listed in 3.5.2(a)–(d) for the regular convolution, namely *commutativity, associativity, distributivity,* and *commutativity of scalar multiplication and convolution* also apply to the cyclical convolution.

(b) *Shift property.* Define the *cyclical back shift operator* σ in the discrete-time case by

$$(\sigma^\theta x)(n) = x((n + \theta) \bmod N), \qquad n \in \underline{N},$$

and in the continuous-time case by

$$(\sigma^\theta x)(t) = x((t + \theta) \bmod P), \qquad t \in [0, P).$$

Then, if $x \odot y$ exists,

$$\sigma^\theta(x \odot y) = (\sigma^\theta x) \odot y = x \odot (\sigma^\theta y)$$

for any $\theta \in \underline{N}$ in the discrete-time case and any $\theta \in [0, P)$ in the continuous-time case. ∎

The proof is not difficult.

Cyclical and Regular Convolution

In what follows we formulate two important connections between the regular and the cyclical convolution. The first result is that if one of the operands of a regular convolution is periodic, the result is also periodic (if it exists). one of the sulting signal may be obtained by periodic extension and cyclical convolution. Other result is that regular convolution of finite support signals may be obtain by cyclical convolution. This is an imprtant result for signal processing, because as seen in chapter 9, there exist very efficient numerical algorithms for cyclical convolution.

3.8.7. Summary: Cyclical and regular convolution.

(a) *Regular convolution with a periodic signal.* Suppose that x and y are crete- or continuous-time infinite-time signals such that y is periodic with period and has finite amplitude. Then, if the action $\|x\|_1$ of x is finite, the convolution

$$z = x * y$$

exists and is periodic with period P. Define the finite-time signals, X, Y, and Z follows:

> X is the one-period restriction of the periodic extension x_{per} of x with period,
> Y is the one-period restriction of y, and
> Z is the one-period restriction of z.

Then,

$$Z = X \odot Y.$$

(b) *Convolution of finite support signals by cyclical convolution.* Suppose that and y are discrete- or continuous-time signals defined on the infinite time axis $\mathbb{T}$ such that x, y and their convolution $z = x * y$ all have their support inside the interval $[0, P)$ for some $P \in \mathbb{T}$. Define $\mathbb{T}_P$ as the finite time axis $\mathbb{T}_P = [0, P) \cap \mathbb{T}$. I.e. Z be the restriction of z to $\mathbb{T}_P$, i.e., Z is defined on $\mathbb{T}_P$, and

$$Z(t) = z(t) \qquad \text{for} \quad t \in \mathbb{T}_P.$$

Similarly, X is the restriction of x to $\mathbb{T}_P$ and Y that of y. Then,

$$Z = X \odot Y.$$ ∎

3.8.8. Proof. The proof of (a) for the continuous-time case is given in the introduction to this section. The existence of z follows from 3.5.3(d). The proof for the discrete-time case is similar.

To prove (b), let y_{per} be the periodic extension of y with period P. By (a), $z = x * y_{\text{per}}$ is periodic with period P. Following (a), define X as the one-period restriction of the periodic extension x_{per} of x. Because by assumption x has finite support within the interval $[0, P)$, by 3.8.2(b) X and x coincide on $\mathbb{T}_P$. Hence, X is as given in (b). Again, following (a), define Y as the one-period restriction of the periodic extension y_{per} of y. Because again by assumption y has finite support within $[0, P)$, also Y is as defined in (b). It follows from (a) that $Z = X \odot Y$, where Z is the one-period restriction of z. By the shift property of the convolution,

$$z = x * y_{per} = x * \sum_{k=-\infty}^{\infty} \sigma^{-kP} y = \sum_{k=-\infty}^{\infty} \sigma^{-kP}(x * y) = (x * y)_{per},$$

where $(x * y)_{per}$ is the periodic extension of $x * y$. Because by assumption $x * y$ has its support within $[0, P)$, by 3.8.2(b) Z and $x * y$ coincide on $\mathbb{T}_P$, which proves (b). ∎

Response of Convolution Systems of Periodic Inputs

We are now fully prepared to express the response of an infinite-time linear time-invariant system to a periodic input in terms of *finite-time* signals.

3.8.9. Summary: Response of a convolution system to a periodic input. Suppose that the impulse response h of a discrete- or continuous-time convolution system has finite action.

(a) If the input u is periodic with period P and has finite amplitude the output y exists and is again periodic with period P.

(b) Let Y denote the one-period restriction of the output y, H the one-period restriction of the periodic extension h_{per} of h with period P, and U the one-period restriction of u. Then,

$$Y = H \odot U.$$ ∎

This result is an immediate consequence of 3.8.7(a). It allows obtaining the response of a convolution system to a periodic input by cyclical convolution.

Figure 3.37 shows how the response to periodic inputs on the one hand follows by regular convolution, and on the other by cyclical convolution. Figure 3.38 is the specialization of the diagram of Fig. 3.37 when the input is the "unit periodic input" Δ_{per} or δ_{per}, successively defined by

$$\Delta_{per}(n) = \sum_{k=-\infty}^{\infty} \Delta(n + kN), \qquad n \in \mathbb{Z},$$

and

$$\delta_{per}(t) = \sum_{k=-\infty}^{\infty} \delta(t + kP), \qquad t \in \mathbb{R}.$$

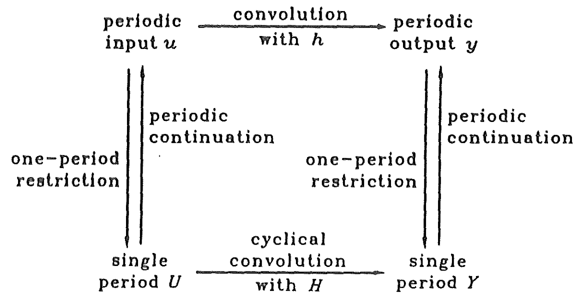


Figure 3.37 The response of a convolution system to a periodic input may be found by regular or by cyclical convolution.

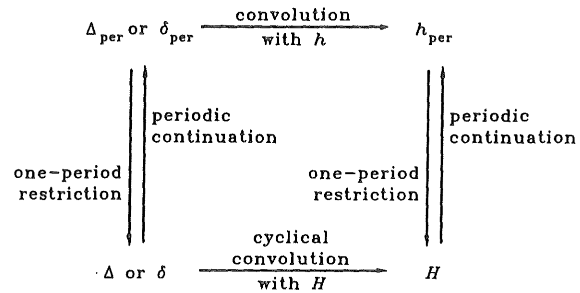


Figure 3.38 Response of a convolution system to the unit periodic input Δ_{per} or δ_{per}.

The result summarized in 3.8.9 shows that the response of a convolution system to a periodic input with a given, fixed period may be obtained by the cyclical convolution

$$Y = H \odot U.$$

This cyclical convolution (with H given) may be thought of as the IO map of a finite-time system, called a *cyclical convolution system*.

3.8.10. Examples: Response to periodic inputs.

(a) *Exponential smoother*. We consider the exponential smoother with a periodic input u with period 4, one period of which is given by

$$u(0) = U(0) = u(1) = U(1) = 1,$$
$$u(2) = U(2) = u(3) = U(3) = 0.$$

U is the one-period restriction of u. From Example 3.4.2(a) we know that the impulse response h of the exponential smoother is given by

$$h(n) = (1 - a)a^n \mathbb{1}(n), \qquad n \in \mathbb{Z}.$$

To apply 3.8.9 we first compute the periodic extension h_{per} of h with period $P = N$. The impulse response h has finite action if $|a| < 1$. For $0 \leq n < N$ we have

$$\begin{aligned} h_{per}(n) &= \sum_{k=-\infty}^{\infty} h(n - kN) \\ &= (1 - a) \sum_{k=-\infty}^{\infty} a^{n-kN} \mathbb{1}(n - kN) = (1 - a)a^n \sum_{k=-\infty}^{0} a^{-kN} \\ &= \frac{1 - a}{1 - a^N} a^n, \qquad 0 \leq n < N, \qquad n \in \mathbb{Z}. \end{aligned}$$

Plots of the impulse response h and its periodic extension h_{per} with period $N = 4$ are given in Fig. 3.39 for $a = 1/2$. The numerical values are

$$h_{per}(0) = H(0) = \frac{8}{15},$$

$$h_{per}(1) = H(1) = \frac{4}{15},$$

$$h_{per}(2) = H(2) = \frac{2}{15},$$

$$h_{per}(3) = H(3) = \frac{1}{15},$$

with H the one-period restriction of h_{per}. The cyclical convolution $Y = H \odot U$ is the cyclical convolution of the signals $(1, 1, 0, 0)$ and $(8/15, 4/15, 2/15, 1/15)$ that is computed in Example 3.8.5. It follows from 3.8.9 that one period of the periodic output is given by

$$y(0) = Y(0) = \frac{3}{5},$$

$$y(1) = Y(1) = \frac{4}{5},$$

$$y(2) = Y(2) = \frac{2}{5},$$

$$y(3) = Y(3) = \frac{1}{5}.$$

Plots of the input together with the output are given in Fig. 3.40.

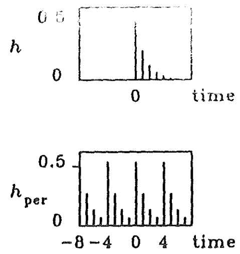


Figure 3.39 Top: the impulse response h of the exponential smoother. Bottom: its periodic extension h_{per} with period 4.

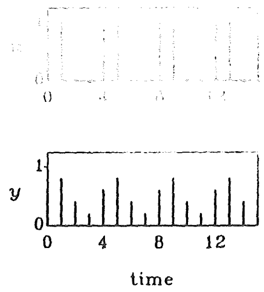


Figure 3.40 Top: periodic input u to the exponential smoother. Bottom: the corresponding output y.

(b) *Echo system.* Consider a continuous-time system whose impulse response is given by

$$h(t) = \sum_{n=0}^{\infty} (1/2)^n \delta(t - nP), \qquad t \in \mathbb{R},$$

with P a positive number. The response of this system to a delta function is a series of regularly spaced delta functions ("echos") with exponentially decreasing coefficients. We study the response of the system to a periodic input whose period P is the same as the time interval between echos. To apply 3.8.9 we first compute the periodic extension h_{per} of the impulse response h. It follows that

$$h_{per}(t) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} (1/2)^n \delta(t + kP - nP), \qquad t \in \mathbb{R}.$$

Interchanging the order of summation and substituting $n - k = m$ we obtain

$$\begin{aligned} h_{per}(t) &= \sum_{n=0}^{\infty} (1/2)^n \sum_{k=-\infty}^{\infty} \delta(t + kP - nP) = \sum_{n=0}^{\infty} (1/2)^n \sum_{m=-\infty}^{\infty} \delta(t - mP) \\ &= 2 \sum_{m=-\infty}^{\infty} \delta(t - mP), \qquad t \in \mathbb{R}. \end{aligned}$$

It follows that the one-period restriction H of h_{per} is

$$H(t) = 2\delta(t), \qquad t \in [0, P).$$

As a result, by 3.8.9 the one-period restriction Y of the response y of the echo system to the periodic input u is

$$Y(t) = (H \odot U)(t) = \int_0^P 2\delta((t-\tau) \bmod P)U(\tau)\,d\tau = 2U(t), \qquad t \in [0, P),$$

with U the one-period restriction of the input u. Hence, if the input is periodic with period P, the output of the echo system is

$$y(t) = 2u(t), \qquad t \in \mathbb{R},$$

independently of the precise behavior of the input. *Exercise:* Obtain the same solution by regular convolution. Show that this interesting result is the effect of superposition of the echos of successive periods. ∎

Response of Convolution Systems to Harmonic Periodic Inputs

We conclude this section with some comments on the response of convolution systems to *harmonic* periodic inputs. As seen in Chapter 2, the harmonic signal η_f defined on the discrete time axis $\mathbb{Z}$ repeats itself periodically with period N if the frequency f satisfies $f \in \mathbb{Z}(F)$, with $F = 1/N$. By aliasing, we actually only need consider frequencies $f \in \underline{N}(F)$. If defined on the continuous time axis $\mathbb{R}$, the harmonic η_f repeats itself periodically with period P if $f \in \mathbb{Z}(F)$, where $F = 1/P$.

As seen earlier in this section, the response of a discrete- or continuous-time convolution system with impulse response h to a periodic input may be obtained by considering the cyclical convolution system with IO map

$$Y = H \odot U.$$

Y is one period of the output and U one period of the input, while H is the one-period restriction (for the given period) of the periodic extension h_{per} (again with the given period) of the impulse response h.

In the discrete-time case, the response of the cyclical convolution system to the harmonic input $U = \eta_f$, with $f \in \underline{N}(F)$, $F = 1/N$, is

$$\begin{aligned} Y(n) &= \sum_{m=0}^{N-1} H((n-m) \bmod N)u(m) \\ &= \sum_{m=0}^{N-1} H((n-m) \bmod N)e^{j2\pi fm}, \qquad n = 0, 1, \cdots, N-1. \end{aligned}$$

By the substitution $n - m = k$ it easily follows that

$$Y(n) = \hat{H}(f)e^{j2\pi fn}, \qquad n = 0, 1, \cdots, N-1, \tag{3}$$

where $\hat{H}$ is defined by

$$\hat{H}(f) = \sum_{n=0}^{N-1} H(n)e^{-j2\pi fn}, \qquad f \in \underline{N}(F).$$

The function $\hat{H}$ is called the *frequency response function* of the cyclical convolution system.

In the continuous-time case a similar result holds. We summarize as follows.

3.8.11. Summary: Response of cyclical convolution systems to harmonic inputs.

The response of the discrete-time cyclical convolution system

$$Y = H \odot U,$$

defined on the time axis $\underline{N}$, to the harmonic input U given by

$$U(n) = \eta_f(n) = e^{j2\pi nf}, \qquad n \in \underline{N},$$

with $f \in \underline{N}(F)$, $F = 1/N$, is

$$Y = \hat{H}(f)\eta_f.$$

The *frequency response function* $\hat{H}$ of the cyclical convolution system is defined by

$$\hat{H}(f) = \sum_{n=0}^{N-1} H(n)e^{-j2\pi fn}, \qquad f \in \underline{N}(F).$$

The response of the continuous-time cyclical convolution system

$$Y = H \odot U,$$

defined on the time axis $[0, P)$, to the harmonic input U given by

$$U(t) = \eta_f(t) = e^{j2\pi ft}, \qquad t \in [0, P),$$

with $f \in \mathbb{Z}(F)$, $F = 1/P$, is

$$Y = \hat{H}(f)\eta_f.$$

The *frequency response function* $\hat{H}$ of the cyclical convolution system is defined by

$$\hat{H}(f) = \int_{0^-}^{P} H(t)e^{-j2\pi ft}\,dt, \qquad f \in \mathbb{Z}(F).$$ ∎

We next investigate what this result means for the response of a regular discrete-time convolution system with impulse response h. By periodic continuation of Y as given by (3) it follows that the response of the regular convolution system to the harmonic input η_f, is

$$y = \hat{H}(f)\eta_f, \qquad f \in \underline{N}(F), \tag{4}$$

with $F = 1/N$. On the other hand, we know from Section 3.7 that the response of the system to the harmonic input $u = \eta_f$ is

$$y = \hat{h}(f)\eta_f,$$

where $\hat{h}$ is the frequency response function of the regular convolution system with IO map $y = h * u$, given by

$$\hat{h}(f) = \sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn}, \qquad f \in \mathbb{R}. \tag{5}$$

Comparison of (4) and (5) shows that

$$\hat{H}(f) = \hat{h}(f), \qquad f \in \underline{N}(F).$$

Indeed, this may be proved directly. Similarly, in the continuous-time case

$$\hat{H}(f) = \hat{h}(f), \qquad f \in \mathbb{Z}(F),$$

$F = 1/P$, where the frequency response function now is given by

$$\hat{h}(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt, \qquad f \in \mathbb{R}.$$

3.8.12. Exercise: Connection between $\hat{h}$ and $\hat{H}$.

Let H be the one-period restriction of the periodic extension h_{per} with period N of a finite action signal h defined on the discrete time axis $\mathbb{Z}$. Define

$$\hat{H}(f) = \sum_{n=0}^{N-1} H(n)e^{-j2\pi fn}, \qquad f \in \underline{N}(F),$$

with $F = 1/N$, and

$$\hat{h}(f) = \sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn}, \qquad f \in \mathbb{R}.$$

Prove that

$$\hat{H}(f) = \hat{h}(f) \qquad \text{for } f \in \underline{N}(F).$$

Let H be the one-period restriction of the periodic extension h_{per} with period P of a finite action signal h defined on the continuous time axis $\mathbb{R}$. Define

$$\hat{H}(f) = \int_0^P H(t)e^{-j2\pi ft}\,dt, \qquad f \in \mathbb{Z}(F),$$

with $F = 1/P$, and

$$\hat{h}(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt, \qquad f \in \mathbb{R}.$$

Prove that

$$\hat{H}(f) = \hat{h}(f) \qquad \text{for } f \in \mathbb{Z}(F). \qquad \blacksquare$$

Figure 3.41 shows the specialization of Fig. 3.37 to periodic harmonic inputs. It clearly demonstrates the equivalence of $\hat{h}$ and $\hat{H}$ on the appropriate set of frequencies.

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\eta_f \arrow[r, "\text{convolution}", "\text{with } h"'] \arrow[d, shift right, "\text{one-period restriction}"'] & \hat{h}(f)\eta_f \arrow[d, shift right, "\text{one-period restriction}"'] \\
\eta_f \arrow[r, "\text{cyclical convolution}", "\text{with } H"'] \arrow[u, shift right, "\text{periodic continuation}"'] & \hat{H}(f)\eta_f \arrow[u, shift right, "\text{periodic continuation}"']
\end{tikzcd}$$

Figure 3.41 The response of convolution systems to a periodic harmonic input with frequency f. In the discrete-time case $f \in \underline{N}(F)$, $F = 1/N$, while in the continuous-time $f \in \mathbb{Z}(F)$, $F = 1/P$.

We conclude with a review of the main result of this section for samled systems

3.8.13. Review: Response of sampled convolution systems to periodic inputs. The periodic extension z_{per} of a sampled signal z defined on the time axis Z(T with period $P \in \mathbb{Z}(T)$ is

$$z_{per}(t) = \sum_{k=-\infty}^{\infty} z(t - kP), \qquad t \in \mathbb{Z}(T).$$

The cyclical convolution $x \odot y$ of two sampled signals x and y given on the time axis $\underline{N}(T)$ is defined as

$$(x \odot y)(t) = T \sum_{\tau \in \underline{N}(T)} x((t - \tau) \bmod NT)\, y(\tau), \qquad t \in \underline{N}(T).$$

Then the one-period restriction Y of the response y of an infinite-time sampled convolution system with impulse response h to a bounded periodic input u with period $P = NT$ is given by

$$Y = H \odot U,$$

where H is the one-period restriction of the periodic extension h_{per} of h and U the one-period restriction of u.

The response of the infinite-time convolution system to the harmonic periodic input η_f, with $f \in \mathbb{Z}(F)$, $F = 1/NT$, is

$$y = \hat{H}(f)\eta_f,$$

where the frequency response function $\hat{H}$ of the sampled cyclical convolution system $Y = H \odot U$ follows from

$$\hat{H}(f) = T \sum_{t \in \underline{N}(T)} H(t)e^{-j2\pi ft}, \qquad f \in \underline{N}(F).$$

$\hat{H}$ is related to the frequency response function

$$\hat{h}(f) = T \sum_{t \in \mathbb{Z}(T)} h(t)e^{-j2\pi ft}, \qquad f \in \mathbb{R},$$

of the infinite-time sampled convolution system $y = h * u$ as

$$\hat{H}(f) = \hat{h}(f), \qquad f \in \underline{N}(F). \qquad \blacksquare$$

3.9 INTERCONNECTIONS OF SYSTEMS

Two or more IO systems may be *interconnected* by arranging that the outputs of some of the systems serve as inputs to other systems. One way of representing such interconnections is by means of a *block diagram*.

We have already met instances of block diagrams in Figs. 1.10, 1.13, and 2.21. The basic element of a block diagram is a box or block as in Fig. 3.42. A block generally has a number of incoming connections and outgoing connections. If the block has K incoming connections, marked, say, $u_1, u_2, \cdots, u_K$, this signifies that the input of the system may be decomposed into K components, $u_1 \in \mathcal{U}_1$, $u_2 \in \mathcal{U}_2, \cdots, u_K \in \mathcal{U}_K$. As a result, the input set of the system is a product set of the form $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_K$. If the block has M outgoing connections, marked, say, $y_1, y_2, \cdots, y_M$, the system output has M components and the output set is also a product set, $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_M$.

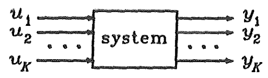


Figure 3.42 Basic element of a block diagram.

A block diagram usually comprises several blocks. In the block diagram it is graphically indicated which component outputs of each subsystem are connected to which component inputs of other subsystems. A connection is allowed only if the associated component output set is contained in the associated component input set. The input to the interconnected (overall) system consists of all unattached incoming connections taken together, while the output of the interconnected (overall) system consists of all unattached outgoing connections. The following example clarifies this.

3.9.1. Example: A simple interconnection. In the block diagram of Fig. 3.43, system 2 has the input u with components u_1 and u_2 and a single output y_1. System 1 has the input v_1 and output z with components z_1 and z_2. Component output z_1 of system 1 is connected to component input u_2 of system 2. The overall input to the interconnected system is (u_1, v_1) and the overall output is (y_1, z_2). If the individual systems are IOM systems with IO maps represented in the form

$$y_1 = \phi(u_1, u_2)$$

and

$$z_1 = \psi_1(v_1),$$
$$z_2 = \psi_2(v_1),$$

the IO map of the interconnected system is given by

$$y_1 = \phi(u_1, \psi_1(v_1)),$$
$$z_2 = \psi_2(v_1).$$

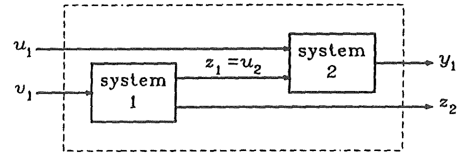


Figure 3.43 Example of a block diagram. ∎

Some common subsystems have special graphical representations. Two well-known ones are depicted in Fig. 3.44. The system on the left is a *branch*. It has input u and two component outputs y_1 and y_2, and its IO map is specified by

$$y_1 = u,$$
$$y_2 = u.$$

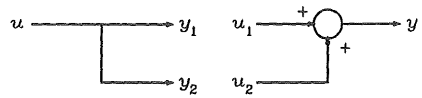


Figure 3.44 Two common subsystems. Left: branch. Right: adder.

The system on the right in Fig. 3.44 is an *adder*. It has two component inputs u_1 and u_2 that belong to the same linear space. The adder has output y given by

$$y = u_1 + u_2.$$

The plus signs may be replaced with minus signs with obvious corresponding changes in the definition of the system.

Series and Parallel Connections

Two well-known connections of IOM systems are the *series* or *cascade connection* and the *parallel connection* as shown in Fig. 3.45. The parallel connection has a branch as well as an adder.

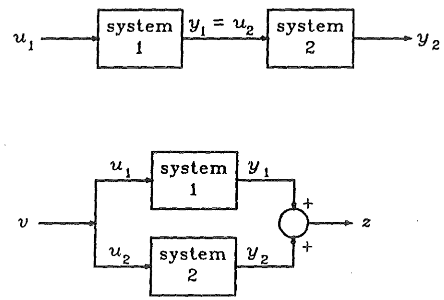


Figure 3.45 Two well-known interconnections of two systems. Top: series connection. Bottom: parallel connection.

In what follows, some facts about the series connection of two IOM systems are collected.

3.9.2. Summary: Series connection.

(a) *IO map*. The series connection of two IOM systems consisting of a system with IO map $\phi_1: \mathcal{U}_1 \to \mathcal{Y}_1$ followed by a system with IO map $\phi_2: \mathcal{U}_2 \to \mathcal{Y}_2$, is well-defined if $\mathcal{Y}_1 \subset \mathcal{U}_2$. It has the IO map $\phi: \mathcal{U}_1 \to \mathcal{Y}_2$ given by

$$\phi = \phi_2 \circ \phi_1,$$

where $\circ$ denotes map composition.

(b) *Impulse response*. If the systems are discrete- or continuous-time convolution systems with impulse responses h_1 and h_2, respectively, the series connection again is a convolution system with impulse response

$$h = h_2 * h_1.$$

(c) *Frequency response function*. If the systems are discrete- or continuous-time convolution systems with frequency response functions $\hat{h}_1$ and $\hat{h}_2$, respectively, the series connection has the frequency response function

$$\hat{h} = \hat{h}_2 \hat{h}_1.$$ ∎

Figure 3.46 illustrates the results. Their proof is simple:

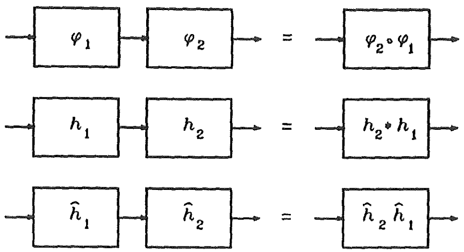


Figure 3.46 Properties of the series connection.

3.9.3. Proof.

(a) By substitution it follows that $y_2 = \phi_2(u_2) = \phi_2(\phi_1(u_1)) = (\phi_2 \circ \phi_1)(u_1)$, so that $\phi = \phi_2 \circ \phi_1$.

(b) Substituting $u_2 = y_1 = h_1 * u_1$ into $y_2 = h_2 * u_2$ it follows by the associativity of the convolution that $y_2 = h_2 * (h_1 * u_1) = (h_2 * h_1) * u_1$. Hence, the series connection is a convolution system with impulse response $h = h_2 * h_1$.

(c) If $u_1 = \eta_f$ (both in the discrete- and the continuous-time case) then $u_2 = y_1 = \hat{h}_1(f)\eta_f$, so that $y_2 = \hat{h}_2(f)\hat{h}_1(f)\eta_f$. It follows that the series connection has the frequency response function $\hat{h} = \hat{h}_2\hat{h}_1$. ∎

For the parallel connection we have what follows.

3.9.4 Summary: Parallel conncetions

(a) IO map. The parallel connection of two IOM systems consisting of a system whith IO map phi1: U --> Y in parallel whith a system whith IO map phi2: U --> Y, well-difined if Y is a linear space, and has the IO map: phi: U --> Y giben by:

$$\phi = \phi_1 + \phi_2.$$

Here, $\phi_1 + \phi_2$ is the map defined by $(\phi_1 + \phi_2)(u) = \phi_1(u) + \phi_2(u)$ for all $u \in \mathcal{U}$

(b) *Impulse response*. If the systems are discrete- or continuous-time convolution systems with impulse responses h_1 and h_2, the parallel connection again is a convolution system with impulse response

$$h = h_1 + h_2.$$

(c) *Frequency response function*. If the systems are discrete- or continuous-time convolution systems with frequency response functions $\hat{h}_1$ and $\hat{h}_2$, the parallel connection has the frequency response function

$$\hat{h} = \hat{h}_1 + \hat{h}_2.$$ ∎

Figure 3.47 illustrates the results, whose proof is left as an exercise.

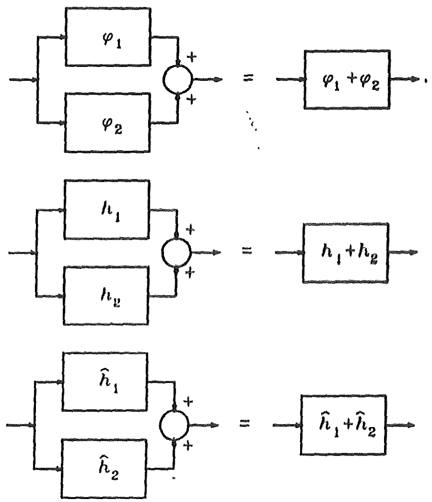


Figure 3.47 Properties of the parallel connection.

3.9.5. Example: Series connection of hifi audio components. As shown in Fig. 3.48, home audio equipment often consists of a number of components connected in series: tuner, cassette deck, record or CD player, followed by an amplifier,

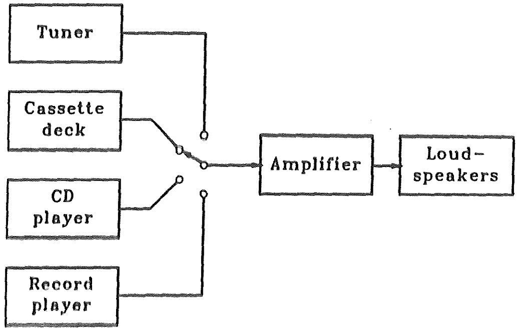


Figure 3.48 Home audio system.

which in turn is connected to the loudspeaker boxes. Each of the three components in the series connection may be characterized by a frequency response function. The frequency response function of the overall system is the product of the three functions. Ideally the overall frequency response function is flat over a wide frequency range, as in Fig. 3.49. The weakest link in the system is the component whose frequency response function starts dropping off soonest on the low- and high-frequency sides. Usually this is the loudspeaker.

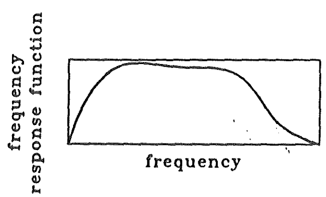


Figure 3.49 Frequency response function of the audio system. ∎

3.9.6. Example: Audio equalizer. Audio amplifiers are often provided with an *equalizer*, by means of which the frequency response characteristics may be adjusted. Its purpose is to compensate for weaknesses in the frequency responses of other components, such as the loudspeakers, or to correct for acoustical peculiarities of the room. An N-band equalizer may be realized as the parallel connection of N band-pass filters as in Fig. 3.50, each with a different center frequency but with partly overlapping bands, and provided with an adjustable gain. A *gain* is a system with IO map $y = ku$, with k a constant. If the ith band-pass filter has frequency response function $\hat{h}_i$ and the corresponding gain is k_i, the overall frequency response function is

$$\hat{h} = k_1\hat{h}_1 + k_2\hat{h}_2 + \cdots + k_N\hat{h}_N,$$

as illustrated in Fig. 3.51.

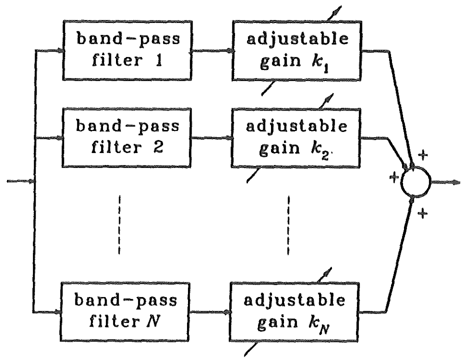


Figure 3.50 N-band equalizer.

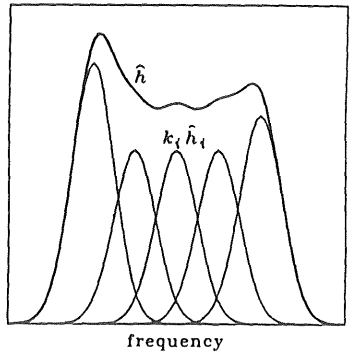


Figure 3.51 Frequency response function of a five-band equalizer. ∎

3.10 PROBLEMS

The first problems for this chapter deal with several elementary notions concerning systems as introduced in Section 3.2.

3.10.1. Adder. A calculating device accepts two real numbers as input and produces their sum as output. Describe this as an input-output mapping system by specifying the input set $\mathcal{U}$, the output set $\mathcal{Y}$ and the input-output map $\phi: \mathcal{U} \to \mathcal{Y}$.

3.10.2. Sorting. A computer routine sorts alphanumeric input data $u_1, u_2, \cdots, u_N$ alphabetically. Each input item u_i is a word of variable but finite length consisting of a sequence of characters from the ASCII character set $\mathbb{A}$, and sorting is lexicographically according to the order of the ASCII characters. Describe this routine as an IOM system by specifying the input and output set $\mathcal{U}$ and $\mathcal{Y}$ and the IO map $\phi: \mathcal{U} \to \mathcal{Y}$.

3.10.3. Memoryless system. Prove that a memoryless IOM system is non-anticipating.

3.10.4. Time scaler. A time scaler is a continuous-time IOM system with input and output sets $\mathcal{U} = \mathcal{Y} = \mathcal{L}$, whose IO map is defined by

$$y(t) = u(\alpha t), \qquad t \in \mathbb{R},$$

with α a real constant.

(a) Prove that this system is time-invariant if and only if $\alpha = 1$.
(b) Similarly, prove that the system is non-anticipating if and only if $\alpha = 1$.

Linear systems, which form the subject of the next series of problems, are introduced in Section 3.3.

3.10.5. Backward and forward differencer. A *backward differencer* is a discrete-time IOM system with time axis $\mathbb{Z}$ and input and output sets $\mathcal{U} = \mathcal{Y} = \ell$, whose IO map is defined by

$$y(n) = u(n) - u(n-1), \qquad n \in \mathbb{Z}.$$

(a) Prove that this system is linear.
(b) Determine its kernel.
(c) Compute and plot the response of the system to the unit step

$$u(n) = \mathbb{1}(n), \qquad n \in \mathbb{Z}.$$

(d) Compute and plot the response of the system to the ramp signal

$$u(n) = n\mathbb{1}(n), \qquad n \in \mathbb{Z}.$$

(e) Is the system non-anticipating?
(f) Repeat (b), (c), and (e) for the *forward differencer*, which is the system defined by

$$y(n) = u(n+1) - u(n), \qquad n \in \mathbb{Z}.$$

3.10.6. Time scaler. A *time scaler*, as introduced in Problem 3.10.4, is a continuous-time IOM system with input and output sets $\mathcal{U} = \mathcal{Y} = \mathcal{L}$, whose IO map is defined by

$$y(t) = u(\alpha t), \qquad t \in \mathbb{R},$$

with α a real constant.

(a) Prove that this system is linear.
(b) Determine the kernel of the system.
(c) Compute and plot the response of the system to a step input

$$u(t) = \mathbb{1}(t), \qquad t \in \mathbb{R}.$$

Distinguish various values of α.

3.10.7. Amplitude modulator. An *amplitude modulator* is a continuous-time system with input and output sets $\mathcal{U} = \mathcal{Y} = \mathcal{L}$ and IO map characterized by

$$y(t) = u(t) \cdot \sin(2\pi f_o t), \qquad f \in \mathbb{R},$$

with f_o a fixed real constant.

(a) Prove that the system is linear and time-varying.
(b) What is the kernel of the system?
(c) Is the system non-anticipating?

3.10.8. Linearization of a "robo arm" figure 3.52 shows a much simplified "robot arm", consisting of a point mass m at the end of a weightless inflexible rod length l that rotates about a fixed pivot O. the input to the system is an extern trque ι while its output is the angle phi the rod makes whith the vertical. Gavity exerts vetical force mg, whih g the acceleration of gravity, on te mass. Since the component of this force perpendicular to the rod is $mg \sin(\phi)$, gravitation results i a torque $mgl \sin(\phi)$ on the rod. Thus, from Newton's law we obtain the equation

$$J\frac{d^2\phi(t)}{dt^2} = T_{\text{total}}(t) = u(t) - mgl \sin[\phi(t)], \qquad t \geq t_o.$$

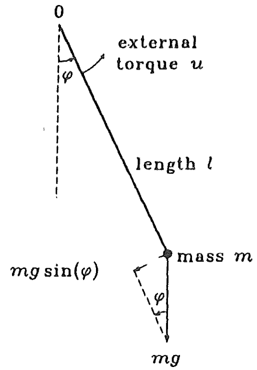


Figure 3.52 Robot arm.

Here, T_{total} is the total torque exerted on the arm, and $J = ml^2$ the moment of inertia of the arm. Dividing by J and substituting $J = ml^2$ we obtain after rearrangement

$$\frac{d^2\phi(t)}{dt^2} + \frac{g}{l}\sin[\phi(t)] = \frac{1}{ml^2}u(t), \qquad t \geq t_o.$$

Consider an equilibrium solution corresponding to a constant input torque $u(t) = U_o$, $t \geq t_o$.

(a) Show that there exists a constant equilibrium solution $\phi(t) = \Phi_o$, $t \geq t_o$ if and only if $|U_o| \leq mgl$. Also show that for each U_o in this range there in fact exist *two* equilibrium solutions. Clarify these geometrically.
(b) Show that for a given equilibrium solution (U_o, Φ_o) the variational system is described by the linear differential equation

$$\frac{d^2\tilde{\phi}(t)}{dt^2} + \frac{g}{l}\cos(\Phi_o)\tilde{\phi}(t) = \frac{1}{ml^2}\tilde{u}(t), \qquad t \geq t_o.$$

Hint: See 3.3.9.

The purpose of the following two problems is to identify a number of basic system properties.

3.10.9. A discrete-time system. Consider the discrete-time system with input signal range $U = \mathbb{R}$ and output signal range $Y = \mathbb{B} = \{0, 1\}$, whose input u and output y are related by

$$y(n) = \begin{cases} 0 \text{ if } \sum_{k=-\infty}^{n} u(k) < 100, \\ 1 \text{ otherwise,} \end{cases} \qquad n \in \mathbb{Z}.$$

Is this system

(a) an IOM system,
(b) linear,
(c) time-invariant,
(d) non-anticipating?

3.10.10. A continuous-time system. Consider a continuous-time system whose input and output sets are subsets of $\mathcal{L}$, and whose input u and output y are related by

$$\dot{y}(t) + y(t) = u(t^2), \qquad t \in \mathbb{R}.$$

Is this system

(a) an IOM system,
(b) linear,
(c) time-invariant?

We continue with some problems on convolution systems, defined in Section 3.4.

3.10.11. Backward and forward differencer. The backward differencer introduced in Problem 3.10.5 is a discrete-time system described by

$$y(n) = u(n) - u(n-1), \qquad n \in \mathbb{Z}.$$

(a) Prove that this IOM system is linear and time-invariant, and hence is a convolution system.
(b) Determine and plot the impulse response of the system.
(c) Also determine the impulse response of the forward differencer, described by

$$y(n) = u(n+1) - u(n), \qquad n \in \mathbb{Z}.$$

3.10.12. Finite and infinite summer. A *finite summer* is a discrete-time system with input and output sets $\mathcal{U} = \mathcal{Y} = \ell$, and IO map defined by

$$y(n) = u(n) + u(n-1) + \cdots + u(n-M+1), \qquad n \in \mathbb{Z}.$$

with M a fixed natural number.

(a) Prove that the system is linear and time-invariant, and hence is a convolution system.
(b) Determine and plot the impulse response of the system.
(c) Determine and plot its step response.
(d) Repeat (b) and (c) for the *infinite summer*, which follows by taking $M = \infty$.

3.10.13. Continuous-time sliding window averager. A continuous-time *sliding window averager* is a system whose input u and output y are continuous-time complex-valued signals that are related by

$$y(t) = \frac{1}{T_1 + T_2} \int_{t-T_1}^{t+T_2} u(\tau)\, d\tau, \qquad t \in \mathbb{R},$$

with T_1 and T_2 nonnegative real numbers such that $T_1 + T_2 \neq 0$.

(a) Prove that this relation can be expressed as a continuous-time convolution $y = h * u$ and determine the impulse response h.
(b) Suppose that the input u is a unit step, that is, $u(t) = \mathbb{1}(t)$, $t \in \mathbb{R}$. Compute the output y.
(c) Use 3.4.3 to establish under what conditions on T_1 and T_2 the system is nonanticipating.

3.10.14. Step response of a continuous-time convolution system. Prove that the step response s and the impulse response h of a continuous-time convolution system are related as

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau, \qquad h(t) = \frac{ds(t)}{dt}, \qquad t \in \mathbb{R}.$$

3.10.15. Discrete-time convolution system with given step response. A discrete-time convolution system has the step response

$$s(n) = n\mathbb{1}(n), \qquad n \in \mathbb{Z}.$$

(a) What is the impulse response of the system?
(b) Determine the response y of the system to the input

$$u(n) = \begin{cases} 0 & \text{for } n < 0, \\ 1 & \text{for } n = 0, \\ -1 & \text{for } n = 1, \\ 0 & \text{for } n \geq 2, \end{cases} \qquad n \in \mathbb{Z}.$$

Plot both u and y.

3.10.16. Continuous-time convolution system with given step response. The step response of a continuous-time convolution system is given by

$$s(t) = \begin{cases} 1 & \text{for } a \leq t < b, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{R},$$

with a and b real constants such that $a < b$.

(a) Determine the IO map of the system.
(b) For which values of a and b is the system non-anticipating?

3.10.17. Continuous-time system with a given IO pair. A continuous-time convolution system has the IO pair (u°, y°) of Fig. 3.53.

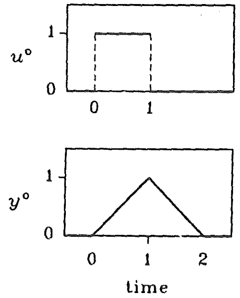


Figure 3.53 An input-output pair of a convolution system.

(a) Find the step response of the system. *Hint:* Use linearity and time-invariance to determine the response of the system to the input u given by $u(t) = u^\circ(t) + u^\circ(t-1) + u^\circ(t-2) + \cdots$, $t \in \mathbb{R}$.
(b) Determine and plot the impulse response of the system.

The next series of problems concern convolution, as discussed in Section 3.5.

3.10.18. Convolutions. Compute the convolutions $x * y$ of the following signal pairs.

(a) $x(n) = \mathbb{1}(n)$, $n \in \mathbb{Z}$, and $y = x$.
(b) $x = \Delta$ is the unit pulse and y is any discrete-time signal defined on the time axis $\mathbb{Z}$.
(c) $x(t) = \text{rect}(t)$, $t \in \mathbb{R}$, and $y = x$.
(d) $x(t) = e^{\alpha t}\mathbb{1}(t)$, $t \in \mathbb{R}$, and $y(t) = e^{\beta t}\mathbb{1}(t)$, $t \in \mathbb{R}$, with α and β complex numbers. Consider the case $\alpha = \beta$ separately.

3.10.19. Convolutions with generalized functions. Determine the convolution $x * y$ when x and y are the following generalized functions.

(a) $x(t) = \delta(t)$ and $y(t) = t$, $t \in \mathbb{R}$.
(b) $x(t) = \delta^{(1)}(t)$ and $y(t) = t$, $t \in \mathbb{R}$.
(c) $x(t) = \mathbb{1}(t-1)$ and $y(t) = \mathbb{1}(t) + \delta^{(1)}(t-1)$, $t \in \mathbb{R}$.
(d) $x(t) = w_P(t)$ and $y(t) = \text{trian}(t/P)$, $t \in \mathbb{R}$, where w_P is the infinite comb given by

$$w_P(t) = \sum_{n=\infty}^{\infty} \delta(t - nP), \quad t \in \mathbb{R}.$$

P is a positive real number.

3.10.20. Shift property of the convolution. Prove the shift property of both the discrete-time and the continuous-time convolution as stated in 3.5.2(e).

3.10.21. Support of the convolution. Suppose that x and y are discrete - or continuous- t signals (both defined on the same time axis) such that the suport of axis x in contain some interval [a,b] and that of y in some other interval [c, d]. Prove the soment of 3.5.5 that the support of x = y is contained in [a + c, b + d].

3.10.22. Convolution of truncated signals. To convolve signals with infinite supp merically it is necessary to *truncate* them to signals with finite support (i.e., to the signals equal to zero outside an interval of finite length). The result of trun tion is that the convolution is computed incorrectly, but not always over the ent support of the convolution. In what follows x and y are either both continuous-ti signals on the time axis $\mathbb{T} = \mathbb{R}$ or both sampled or discrete-time signals defined the time axis $\mathbb{T} = \mathbb{Z}(T)$.

(a) *Truncation on the right of right one-sided signals.* Suppose that the support of lies inside the interval $[a, b]$ and that of y inside the interval $[c, d]$, with a a c finite and b and d possibly infinite. Let x be truncated on the right to the inte val $[a, B]$, that is, the truncation x_{tr} of x is defined as

$$x_{tr}(t) = \begin{cases} x(t) & \text{for } a \le t \le B, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{T},$$

with $B \le b$. Similarly, let y be truncated to the interval $[c, D]$, with $D \le d$. Prove that the convolution $x_{tr} * y_{tr}$ of the truncated signals x_{tr} and y_{tr} agrees with $x * y$ on the interval $(-\infty, e]$, where

$$e = \min(a + D, c + B).$$

Hint: Write the truncation of x as $x_{tr} = x + x_B$ and that of y as $y_{tr} = y + y_D$, and prove that $x_{tr} * y_{tr} = x * y + \bar{z}$, where $\bar{z}$ has support inside (e, ∞).
(b) *Truncation on the left of left one-sided signals.* Find a result similar to (a) for the effect of truncation on the left of left one-sided signals.
(c) *Truncation on both sides.* If both x and y have two-sided infinite support, truncation of the two signals leads to convolutions that are in error over the entire time axis. If the signals are very small outside the truncation intervals it may be hoped that the truncation errors are also small. Prove that if x has two-sided infinite support but is truncated to the interval $[A, B]$, and y has finite support inside $[c, d]$, the convolution of the truncation x_{tr} of x with the (untruncated) signal y coincides with $x * y$ on $[A + d, B + c]$.

3.10.23. Convolution as matrix multiplication. If $u \in \ell$ is a discrete-time signal such that $u(n) = 0$ for $n < 0$, and N is any nonnegative integer, define the column vector $\vec{u}$ and the matrix M_u as follows:

$$\vec{u} = \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \cdots \\ u(N-1) \end{bmatrix}, \quad M_u = \begin{bmatrix} u(0) & 0 & 0 & \cdots & 0 \\ u(1) & u(0) & 0 & \cdots & 0 \\ u(2) & u(1) & u(0) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u(N-1) & u(N-2) & u(N-3) & \cdots & u(0) \end{bmatrix}.$$

The matrix M_u has the property that the elements along each diagonal are all equal. Such a matrix is called a *Toeplitz* matrix. Because moreover all elements above the main diagonal are zero, the matrix is also *lower triangular*.

Let x and y be discrete-time signals in ℓ such that $x(n) = y(n) = 0$ for $n < 0$. By 3.5.3(b) their convolution $z = x * y$ exists.

(a) Show that $z(n) = 0$ for $n < 0$.
(b) Prove that

$$\vec{z} = M_x\vec{y} = M_y\vec{x}.$$

(c) Suppose that x and y are given by

$$x(n) = \begin{cases} 0 & \text{for } n < 0, \\ 1 & \text{for } n \geq 0, \end{cases} \qquad n \in \mathbb{Z}.$$

$$y(n) = \begin{cases} 0 & \text{for } n < 0 \text{ and } n \geq 3, \\ 1 & \text{for } 0 \leq n < 3, \end{cases} \qquad n \in \mathbb{Z}.$$

Use (b) to compute $z = x * y$ on the time interval $0 \leq n \leq 5$, $n \in \mathbb{Z}$.

> Otto Toeplitz (1881–1940) was a German mathematician who emigrated to Palestine in 1938. He contributed to the theory of quadratic forms.

The following problems concern BIBO stability as introduced in Section 3.6.

3.10.24. "Impulsive" systems. Consider the continuous-time system with input set $\mathcal{U} \subset \mathcal{L}$ and output set $\mathcal{Y} \subset \mathcal{L}$, whose IO map is of the form

$$y(t) = \int_{-\infty}^{\infty} h^o(t-\tau)u(\tau)\,d\tau + \sum_{i\in\mathbb{Z}} a_i u(t - t_i), \qquad t \in \mathbb{R}.$$

Here, $h^o \in \mathcal{L}$ is a given function, while the a_i, $i \in \mathbb{Z}$, are real or complex coefficients and the t_i, $i \in \mathbb{Z}$, distinct real time instants.

(a) Prove that the system is linear and time-invariant.
(b) Determine the impulse response of the system.
(c) Prove that the system is non-anticipating if and only if $h^o(t) = 0$ for $t < 0$ and $a_i = 0$ for $t_i < 0$.
(d) Prove that the system is BIBO stable if and only if both h^o and a have finite action, where a is the sequence $\{\cdots, a_{-1}, a_0, a_1, a_2, \cdots\}$.

3.10.25. Echo system. A continuous-time system defined on the infinite time axis $\mathbb{T} = \mathbb{R}$ is described by the IO relation

$$y(t) = u(t) + \alpha y(t - \theta), \qquad t \in \mathbb{R},$$

with θ a fixed positive real constant, and α a real constant. The first term is the direct effect of the input on the output, while the second represents an *echo*.

(a) Show that this system is linear and time-invariant, and find its impulse response.
(b) For which values of α is the system BIBO stable?

3.10.26. BIBO stability. Verify whether the following systems are BIBO stable. If the system is not BIBO stable, show an example of a bounded input that results in an unbounded output.

(a) The backward and forward differencers of Problem 3.10.5.
(b) The finite summer of Problem 3.10.12.
(c) The infinite summer of Problem 3.10.12.
(d) The continuous-time sliding window averager of Problem 3.10.13.
(e) The system of Problem 3.10.15.
(f) The system of Problem 3.10.16.

3.10.27. Output maximization. Consider a continuous-time non-anticipating convolution system with impulse response h such that

$$h(t) = 0 \qquad \text{for} \quad t \geq T,$$

with T a fixed positive time.

(a) Determine an input u with $|u(t)| \leq 1$ for all $t \in \mathbb{R}$ such that $|y(T)|$ is maximal.
(b) Show that the maximal value of $|y(T)|$ equals the action $\|h\|_1$ of h.

Hint: See 3.6.3.

The following problems deal with the frequency response of convolution systems, as discussed in Section 3.7.

3.10.28. Frequency response functions. Determine whether the following systems have a frequency response function. If the system has a frequency response function, calculate it and sketch its magnitude and phase as a function of frequency.

(a) The backward differencer of Problem 3.10.5.
(b) The forward differencer of Problem 3.10.5.
(c) The finite summer of Problem 3.10.12.
(d) The infinite summer of Problem 3.10.12.
(e) The continuous-time delay, which is the continuous-time system described by

$$y(t) = u(t - \theta), \qquad t \in \mathbb{R},$$

with θ a positive real constant.

3.10.29. Response to real harmonic inputs.

(a) Given the frequency response function of the backward differencer as obtained in Problem 3.10.28(a), determine the response of the system to the real harmonic input

$$u(n) = u_{\max}\cos(2\pi f_o n), \qquad n \in \mathbb{Z}.$$

Are there any frequencies f_o for which the output of the system is identical to zero? Answer the same questions for the forward differencer.

(b) Determine the response of the continuous-time system with frequency response function

$$\hat{h}(f) = \frac{1}{(1 + j2\pi f)^2}, \qquad f \in \mathbb{R},$$

to the real harmonic input

$$u(t) = \sin(t), \qquad t \in \mathbb{R}.$$

3.10.30. Symmetry properties of the frequency response function. Let $\hat{h}$ be the frequency response function of a *real* discrete- or continuous-time linear time-invariant system, i.e., a system whose impulse response h is a real-valued function. Prove the symmetry properties of 3.7.6:

(i) *Conjugate symmetry:* $\hat{h}(-f) = \overline{\hat{h}(f)}$ for all $f \in \mathbb{R}$.
(ii) *Evenness of the magnitude:* $|\hat{h}|$ is an even function.
(iii) *Oddness of the phase:* $\arg(\hat{h})$ is an odd function.

The next series of problems relate to cyclical convolution, treated in Section 3.8.

3.10.31. Periodic extension. Determine the periodic extension with period P of the following continuous-time signals:

(a) The one-sided exponential signal

$$x(t) = e^{-t}\mathbb{1}(t), \qquad t \in \mathbb{R}.$$

(b) The rectangular pulse

$$x(t) = \begin{cases} 1 & \text{for } 0 \le t < a, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{R},$$

with (b.1) $0 < a \le P$, and (b.2) $P < a \le 2P$.

3.10.32. Cyclical convolution. Determine the cyclical convolution of the following pairs of signals x and y:

(a) The discrete-time signals defined on the time axis $\underline{8}$ given by $x = (1, 1, 1, 1, 0, 0, 0, 0)$ and $y = (1, 1, 0, 0, 0, 0, 0, 0)$.
(b) The continuous-time signals defined on the time axis $[0, 1)$ by

$$x(t) = y(t) = \begin{cases} 1 & \text{for } 0 \le t < 1/2, \\ 0 & \text{for } 1/2 \le t < 1. \end{cases}$$

3.10.33. Shift property of the cyclical convolution. Prove the shift property 3.8.6(b) of the continuous-time cyclical convolution.

3.10.34. Response of RC network to the periodic rectangular pulse. If RC = impulse response of the RC network is given b

h(t) = e^-t1(t), t E R.

(a) compute the periodic extension hper of h with period

(b) use cyclical convolution to determine the response of the RC network to th
riodic rectangular pulse u, one period of which is given by

$$u(t) = \begin{cases} 1 & \text{for } 0 \le t < 1/2, \\ 0 & \text{for } 1/2 \le t < 1. \end{cases}$$

The final problems concern interconnections of systems, dealt with in Section 3.9.

3.10.35. An interconnection. Three linear time-invariant systems with impulse response h_1, h_2 and h_3 are interconnected as in Fig. 3.54. The systems are either all three discrete-time or all three continuous-time.

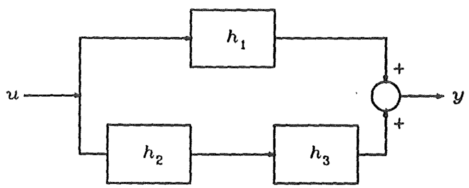


Figure 3.54 An interconnection of three linear time-invariant systems.

(a) Determine the impulse response of the interconnection.
(b) Suppose that the systems have frequency response functions $\hat{h}_1$, $\hat{h}_2$, and $\hat{h}_3$, respectively. Determine the frequency response function of the interconnection.

3.10.36. Series connection of two differencers. Consider the series connection of a backward and a forward differencer (see Problem 3.10.5), as in Fig. 3.55. Determine

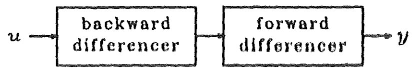


Figure 3.55 Series connection of two differencers.

(a) the impulse response
(b) the frequency response function

of the series connection.

3.10.37. Series connection. Consider two IOM systems S_1 and S_2 connected in series as in Fig. 3.56. We denote the series connection as $S_2 \circ S_1$. Prove the following four statements. In each case give an example to show that the *converse* statement is not true.

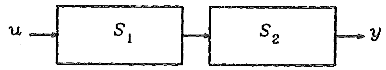


Figure 3.56 Series connection of two IOM systems.

(a) S_1 is linear and S_2 linear $\Rightarrow S_2 \circ S_1$ is linear.
(b) S_1 is time-invariant and S_2 is time-invariant $\Rightarrow S_2 \circ S_1$ is time-invariant.
(c) S_1 is non-anticipating and S_2 is non-anticipating $\Rightarrow S_2 \circ S_1$ is non-anticipating.
(d) S_1 is memoryless and S_2 is memoryless $\Rightarrow S_2 \circ S_1$ is memoryless.

3.11 COMPUTER EXERCISES

The computer exercises for Chapter 3 serve to introduce various elementary ways to handle linear time-invariant systems numerically. The first exercises deal with convolution.

3.11.1. Convolution. (Compare 3.10.18.) Compute and plot the convolutions $x * y$ of the following signals. Identify the effect of truncation, which may make some of the results partially incorrect. For continuous-time convolutions be sure to choose the sampling interval small enough.

(a) $x(n) = \mathbb{1}(n)$, $n \in \mathbb{Z}$, and $y = x$.
(b) $x = \Delta$ is the unit pulse and y is defined by $y(n) = a^{|n|}$, $n \in \mathbb{Z}$, with $a = 0.9$.
(c) $x(t) = \text{rect}(t)$, $t \in \mathbb{R}$, and $y = x$.
(d) $x(t) = e^{\alpha t}\mathbb{1}(t)$ and $y(t) = e^{\beta t}\mathbb{1}(t)$, $t \in \mathbb{R}$, with $\alpha = \beta = -1$.

3.11.2. Convolutions with δ-functions. Generate the sampled approximations d, d_1, and d_2 of the δ-function and its first two derivatives as in 2.7.10.

(a) Check the equalities $\delta * \delta' = \delta'$ and $\delta' * \delta' = \delta''$ numerically. Explain any discrepancies.
(b) Define the function ϕ as in 2.7.10(c), and compute the convolutions $\delta * \phi$, $\delta' * \phi$, and $\delta'' * \phi$ numerically. Verify the results and explain any discrepancies.

Numerical convolution.
Numerical convolution involves several kinds of errors.

Truncation errors. For numerical computation time signals with infinite support necessarily need be *truncated* to signals with finite support. Convolving truncated signals leads to truncation errors. In Problem 3.10.22 it is shown that if the signals are one-sided or one of the signals has finite support the convolution of the signals is correct inside a well-defined interval, and hence is free of truncation errors on this interval. Outside this interval the truncation errors may be expected to be small if the signals are small outside their truncation intervals.

Sampling errors. To compute the convolution of continuous-time signals they need be sampled, so that the convolution integral may be replaced by an approximating convolution sum. To keep the resulting error small the sampling interval should be chosen sufficiently small.

Rounding errors. On top of the truncation and sampling errors there are rounding errors. They may be kept small by using sufficient numerical precision.

3.11.3. Interpolation. As discussed in Section 2.3, the interpolation of a sampled signal x^*, defined on the time axis $\mathbb{Z}(T)$, using the interpolation function i is the continuous-time signal given by

$$x(t) = \sum_{n \in \mathbb{Z}} x^*(nT)\, i\left(\frac{t - nT}{T}\right), \qquad t \in \mathbb{R}.$$

Define the continuous-time signals $x^\dagger$ and i_T by

$$x^\dagger(t) = \sum_{n \in \mathbb{Z}} x^*(nT)\delta(t - nT), \qquad t \in \mathbb{R},$$

$$i_T(t) = i\left(\frac{t}{T}\right), \qquad t \in \mathbb{R}.$$

(a) Prove that the interpolated signal x may be expressed as

$$x = x^\dagger * i_T.$$

(b) In Fig. 2.20 it is shown how a discrete-time signal is interpolated with three different interpolating functions. Reproduce this figure taking $T = 1/8$. *Hint:* Define a discrete time axis with increment `inc` $\ll T$ to approximate the continuous time axis, and approximate the signal $x^\dagger$ as

$$x^\dagger(t) = \begin{cases} \dfrac{x^*(t)}{\texttt{inc}} & \text{for } t \in \mathbb{Z}(T), \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{Z}(\texttt{inc}).$$

It is convenient to choose T as an integral multiple of `inc`.

3.11.4. Response of the sliding window averager. In 3.2.8(b) the discrete-time sliding window averager is described as a system whose output y corresponding to the input u is given by

$$y(n) = \frac{1}{N + M + 1} \sum_{m=-M}^{N} u(n + m), \qquad n \in \mathbb{Z},$$

with N and M nonnegative integers.

(a) Define a function "slide" that implements the IO map, that is, if u is a discrete-time signal, then $y = \text{slide}(u, M, N)$ is the output of the sliding window averager with parameters M and N. *Hint:* Write

$$\sum_{m=-M}^{N} u(n + m) = \sum_{k=n-M}^{n+N} u(k) = \sum_{k=-\infty}^{n+N} u(k) - \sum_{k=-\infty}^{n-M-1} u(k)$$
$$= s(n + N) - s(n - M - 1), \qquad n \in \mathbb{Z},$$

where s is the "summed" signal defined by

$$s(n) = \sum_{k=-\infty}^{n} u(k), \qquad n \in \mathbb{Z}.$$

Use a suitable standard function to compute the summed signal.

(b) Determine and plot the impulse response of the averager by computing its response to the unit pulse $u = \Delta$. Take $N = 10$ and $M = 5$. *Hint:* Note that the result is in error near the end of the time axis owing to truncation. The error may be corrected by restricting the impulse response as computed to a slightly smaller interval.

(c) Determine and plot the response of the system to a step and to a ramp

(c.1) directly using the IO map
(c.2) by convolution with the impulse response.

Hint: Again there are differences near the end of the time axis. Explain and correct.

3.11.5. Response of the RC network. In 3.4.2(b) it is found that if the output y of the RC network is the voltage v_C across the capacitor, the initially-at-rest network may be described as the continuous-time convolution system

$$y = h * u,$$

where the impulse response h is given by

$$h(t) = \frac{1}{RC} e^{-t/RC} \mathbb{1}(t), \qquad t \in \mathbb{R}.$$

Suppose that rather than the voltage $y = v_C$ across the capacitor we take the voltage $z = v_R$ across the resistor as output. Then since by Kirchhoff's voltage law $u = v_C + v_R$ or $u = y + z$, the IO map of the system now is

$$z = u - y = u - h * u.$$

We may rewrite this as the convolution system

$$z = g * u,$$

whose impulse response g is given by

$$g = \delta - h.$$

Take $RC = 1$, and let u be the rectangular input as in 3.5.1(b) with $a = 2$.

(a) Determine and plot the response of the voltage y across the capacitor to this input by convolution of the input with the impulse response h.

(b) Determine and plot the response of the voltage z across the resistor to this inp by convolution of the input with the impulse response g. Verify that $y + z =$ Explain any discrepancies.

(c) Compute and plot the step response of each of the two systems considered.

Discrete-time approximation of the δ-function.

Suppose that a discrete time axis with small increment `inc` is used to approximate a continuous time axis. Then the δ-function may be approximated by the discrete-time signal d given by

$$d(t) = \begin{cases} \dfrac{1}{\text{inc}} & \text{for } t = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{Z}(\text{inc}).$$

3.11.6. Frequency response of the exponential smoother. According to 3.7.5(a) the frequency response function of the exponential smoother is

$$\hat{h}(f) = \frac{1 - a}{1 - ae^{-j2\pi f}}, \qquad f \in \mathbb{R}.$$

(a) Let $a = 0.5$. Reproduce the plots of the magnitude and phase of $\hat{h}$ of Fig. 3.23. Include in the same figure plots for $a = 0.2$ and $a = 0.9$, and comment on the differences.

(b) Again let $a = 0.5$, and suppose that the input to the exponential smoother is the real harmonic signal

$$u(t) = \cos(2\pi f t), \qquad t \in \mathbb{Z},$$

with frequency $f = 0.25$. Use 3.7.7 to determine the response to this input. Plot both the harmonic input and the response.

3.11.7. Frequency response of the RC network. According to 3.7.5(b) the frequency response function of the RC network is given by

$$\hat{h}(f) = \frac{1}{1 + RCj2\pi f}, \qquad f \in \mathbb{R}.$$

Assume that $RC = 1$.

(a) Reproduce the magnitude and phase plots of $\hat{h}$ as shown in Fig. 3.24.

(b) Use 3.7.7 to compute the response of the RC network to the real harmonic input

$$u(t) = \cos(2\pi f_o t), \qquad t \in \mathbb{R},$$

for $2\pi f_o = 1/RC$. Plot the input u and the corresponding output y.

(c) Use 3.7.7 to compute the response of the RC network to the composite real harmonic input u given by

$$u(t) = \cos(2\pi f_o t) + \tfrac{1}{4}\cos(2\pi f_1 t), \qquad t \in \mathbb{R},$$

with f_o as in (b), and $2\pi f_1 = 3/RC$. Plot both the input u and the output y. *Hint:* Use the linearity of the system.

3.11.8. Series connection of two differencers. According to 3.10.5 a (backward) differencer is a discrete-time IOM system whose IO map is specified by

$$y(n) = u(n) - u(n-1), \qquad n \in \mathbb{Z}.$$

(a) It is simple to program the IO map. Determine the impulse response of the system by determining its response to the unit pulse Δ.
(b) Use 3.9.2 to determine the impulse response of the series connection of the differencer with itself in the following two ways:
  (b.1) by map composition, that is, by determining the response of the second system to the response of the first system to the unit pulse;
  (b.2) by convolution of the impulse response of the differencer with itself.
(c) Confirm numerically that the step response of the series connection equals the impulse response of a single differencer. Explain.
(d) Show (theoretically) that the frequency response of a single backward differencer is given by

$$\hat{h}(f) = 1 - e^{-j2\pi f}, \qquad f \in \mathbb{R}.$$

Plot the magnitude and phase of $\hat{h}$ on the frequency interval $[0, \tfrac{1}{2}]$. Why is it sufficient to plot $\hat{h}$ on this interval?
(e) Use 3.9.2 to compute the frequency response function of the series connection of two backward differencers. Plot its magnitude and phase.

4

Difference and Differential Systems

4.1 INTRODUCTION

Most input-output systems dealt with in this text are *difference systems* or *differential systems*. These are input-output systems whose IO relationship consists of a *difference* or *differential* equation. The importance of difference and differential systems stems from the fact that application of the natural laws that govern the behavior of a system over a small time interval often leads to difference equations in the discrete-time case and differential equations in the continuous-time case. The more complex the phenomena are, the higher is the order of the difference or differential equation.

Examples of discrete-time systems described by a difference equation are the exponential smoother of Example 1.2.6, the sliding window averager of Example 3.2.8(b), the savings account of Example 3.2.14, and the national economy of Example 3.2.15. We have also encountered instances of continuous-time systems described by differential equations, such as the RC circuit of Example 1.2.7, the heated vessel of Example 3.2.12, and the moving car of Example 3.2.13. All these systems are described by *first-order* difference and differential equations.

In Section 4.2 we define *difference systems* as discrete-time IO systems whose IO relationship consists of a difference equation involving the input and output. Similarly, *differential systems* are continuous-time systems whose IO relationship takes the form of a differential equation relating input and output. The section includes several examples of difference and differential systems described by difference or differential equations of higher order.

Section 4.3 deals with a number of basic issues. First it is shown how given an input, corresponding outputs follow from the difference and differential equation, given suitable *initial conditions*. Next we consider the *non-anticipativity*, *time-invariance*, and *linearity* of difference and differential systems. Difference and differential systems that are both linear and time-invariant are described by linear difference and differential equations with constant coefficients. The assumption that the initial conditions are zero leads to a *unique* output for any given input. The IO *mapping* system thus defined is called the *initially-at-rest* difference or differential system. Linear time-invariant initially-at-rest difference and differential systems are convolution systems.

The rest of the chapter is concerned with linear time-invariant difference and differential systems. Section 4.4 presents an explicit analysis of the response of linear time-invariant difference and differential systems. First *homogeneous* difference and differential equations are studied. These may easily be solved. Given the general solution of the homogeneous equation together with a *particular* solution we may find the general solution of the nonhomogeneous equation, which gives us all possible outputs corresponding to a given input. In Section 4.5 it is studied how to find the *impulse response* of the initially-at-rest system defined by the difference or differential equation and how to use this to determine the response of the system when not initially at rest.

In Section 4.6 we discuss the *stability* of difference and differential systems. Besides BIBO stability, which was introduced for convolution systems in Section 3.6, we define the notion of *CICO* (converging-input converging-output) stability. The BIBO and CICO stability of constant coefficient difference and differential systems may easily be established from the locations of the *characteristic roots* and the *poles* of the system.

Section 4.7, finally, is devoted to an analysis of the *frequency response* function of difference and differential systems. It may directly be obtained from the coefficients of the difference and differential equation. Its existence depends on the locations of the poles of the system. We, furthermore, consider the *transient* and *steady-state* response to harmonic inputs and show how the frequency response functions of electrical networks may conveniently be obtained by using *impedances*.

4.2 DIFFERENCE AND DIFFERENTIAL SYSTEMS: DEFINITION AND EXAMPLES

Difference systems are discrete-time systems whose IO relation consists of a difference equation. Similarly, differential systems are continuous-time systems whose rule consists of a differential equation. We consider time axes $\mathbb{T}$ that are either *right semi-infinite*, such as $\{n_o, n_o + 1, n_o + 2, \cdots\}$ and $[t_o, \infty)$, or *infinite*, namely, and $\mathbb{R}$.

4.2.1. Definition: Difference and differential systems.

A *difference system* is a discrete-time IO system whose time axis $\mathbb{T}$ is right semi-infinite or infinite, such that any IO pair (u, y) satisfies a difference equation of the form

$$F[y(n), y(n+1), \cdots, y(n+N), u(n), u(n+1), \cdots, u(n+M), n] = 0 \qquad (1)$$

for all $n \in \mathbb{T}$, with N and M nonnegative integers and F a given map $F: \mathbb{C}^{N+M+2} \times \mathbb{T} \to \mathbb{C}$.

A *differential system* is a continuous-time IO system whose time axis $\mathbb{T}$ is right semi-infinite or infinite, such that any IO pair (u, y) satisfies a differential equation of the form

$$F\left[y(t), \frac{dy(t)}{dt}, \cdots, \frac{d^N y(t)}{dt^N}, u(t), \frac{du(t)}{dt}, \cdots, \frac{d^M u(t)}{dt^M}, t\right] = 0 \qquad (1')$$

for all $t \in \mathbb{T}$, with N and M nonnegative integers and F a given map $F: \mathbb{C}^{N+M+2} \times \mathbb{T} \to \mathbb{C}$. ∎

Usually, given an input u to a difference or differential system, there are *many* corresponding outputs y satisfying the difference or differential equation. We shall learn that by imposing extra conditions, in particular *initial conditions*, a unique output y may be established.

The difference equation (1) describing difference systems has been put into a standard form such that the values of the input u and output y at time n and a (finite) number of *later* times $n + 1, n + 2, n + 3, \cdots$, are interrelated.

We assume in the following that for any $n \in \mathbb{T}$ the difference equation (1) may uniquely be solved for $y(n + N)$ in terms of the other arguments. The number N is called the *order* of the difference equation. Also, the difference system is said to have order N. Similarly, we assume that the differential equation (1') may be uniquely solved for $y^{(N)}(t)$ for any $t \in \mathbb{T}$. Again, the differential equation and corresponding system are said to be of *order N*.

4.2.2. Review: Sampled difference systems. Sampled difference systems are sampled systems defined on the semi-infinite time axis $\mathbb{T} = \{t_o, t_o + T, t_o + 2T, \cdots\}$, with $t_o \in \mathbb{Z}(T)$, or the infinite time axis $\mathbb{T} = \mathbb{Z}(T)$. They are described by a difference equation of the form

$$F[y(t), y(t+T), \cdots, y(t+NT), u(t), u(t+T), \cdots, u(t+MT), t] = 0, \qquad t \in \mathbb{T}.$$

We assume that for any $t \in \mathbb{T}$ the difference equation may uniquely be solved for $y(t + NT)$, and call N the *order* of the difference equation and the corresponding IO system. ∎

Examples of Difference and Differential Systems

We have already met several examples of difference and differential systems.

4.2.3. Examples: First-order difference and differential systems.

(a) *Exponential smoother.* The exponential smoother of Example 1.2.6 is described by the difference equation

$$y(n + 1) - ay(n) - (1 - a)u(n + 1) = 0, \qquad n \in \mathbb{T},$$

with time axis $\mathbb{T} = \{n_o, n_o + 1, n_o + 2, \cdots\}$, with n_o an integer. If $n_o = -\infty$, the time axis is $\mathbb{Z}$. The sampled version of the exponential smoother (see 3.7.12 and 4.2.2) is described by the difference equation

$$y(t + T) - ay(t) - (1 - a)u(t + T) = 0, \qquad t \in \mathbb{T},$$

with time axis $\mathbb{T} = \{t_o, t_o + T, t_o + 2T, \cdots\}$, with $t_o \in \mathbb{Z}(T)$. If $t_o = -\infty$, the time axis is $\mathbb{Z}(T)$.

(b) *Savings account.* The savings account of Example 3.2.14 is a difference system described by the difference equation

$$y(n + 1) - (1 + \alpha)y(n) - u(n) = 0, \qquad n = 0, 1, 2, \cdots.$$

(c) *RC network.* The RC circuit, as introduced in Example 1.2.7, is a differential system characterized by the differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) - \frac{1}{RC}u(t) = 0, \qquad t \in [t_0, \infty).$$

(d) *Moving car.* The moving car of Example 3.2.13 is also a differential system, whose differential equation is

$$M\frac{dv(t)}{dt} + Bv^2(t) - cu(t) = 0, \qquad t \in [t_o, \infty).$$

∎

The orders of the systems of 4.2.3 are all one. We next look at some slightly more complex systems.

4.2.4. Examples: Higher-order systems.

(a) *Discrete-time sliding window averager.* The discrete-time sliding window averager of Example 3.2.8(b) is described by the difference equation

$$y(n) - \frac{1}{N + M + 1}\sum_{m=-M}^{N} u(n + m) = 0, \qquad n \in \mathbb{Z},$$

with N and M nonnegative integers. If $M > 0$ this equation is not in the standard form (1), but it may be transformed to it by the simple substitution $n = n' + M$. It follows that

$$y(n' + M) - \frac{1}{N + M + 1}\sum_{m=-M}^{N} u(n' + M + m) = 0, \qquad n' \in \mathbb{Z}.$$

By the change of variable $m' = m + M$ this equation takes the form

$$y(n' + M) - \frac{1}{N + M + 1}\sum_{m'=0}^{N+M} u(n' + m') = 0, \qquad n' \in \mathbb{Z}.$$

This in turn may be rewritten in full as

$$y(n + M) - \frac{1}{N + M + 1}[u(n) + u(n + 1) + \cdots + u(n + N + M)] = 0,$$

for $n \in \mathbb{Z}$, where we drop the prime on n. The system has order M.

(b) *Second-order smoother.* The difference equation

$$y(n) = ay(n - 1) + (1 - a)u(n), \qquad n \in \mathbb{Z},$$

obtained by replacing n with $n - 1$ in the equation for the exponential smoother, shows that the current output $y(n)$ of the smoother is a weighted sum of the immediate past output $y(n - 1)$ and the current input $u(n)$. Inspection of the frequency response function of the smoother, as obtained in 3.7.5(a), shows that it is a low-pass filter. The filter does not cut off high frequencies very sharply, however. We could contemplate including extra terms in the difference equation involving further past outputs and also past inputs, in the hope that this improves the filtering effect. Taking one more past output value and one past input value, for instance, we obtain

$$y(n) = a_1y(n - 1) + a_0y(n - 2) + b_2u(n) + b_1u(n - 1), \qquad n \in \mathbb{Z},$$

with a_1, a_0, b_2, and b_1 constants to be determined. The constants need be selected carefully for the given filtering task, which is a problem we can only deal with later. To obtain the difference equation in the standard form (1) we replace n with $n + 2$ and thus find

$$y(n+2) - a_1 y(n+1) - a_0 y(n) - b_2 u(n+2) - b_1 u(n+1) = 0, \qquad n \in \mathbb{Z}.$$

Comparison with the standard form (1) shows that $N = 2$, so that the order of the system is two. Also, M equals 2.

(c) *RCL circuit*. The input to the RCL circuit of Fig. 4.1 is the voltage u produced by the voltage source. We consider two different choices for the output, namely,

(i) the output is the current i through the circuit, and
(ii) the output is the voltage v_L across the inductor.

The differential equations that describe the system in these two cases may be derived as follows. By Kirchhoff's voltage law,

$$u(t) = v_R(t) + v_C(t) + v_L(t), \qquad t \in \mathbb{R}.$$

> The voltage and current laws are the first two of the four laws found by the German physicist Gustav Robert Kirchhoff (1824–1887.) The other two laws concern the spectral emission of glowing bodies.

For the resistor, capacitor, and inductor we have

$$v_R(t) = Ri(t), \qquad v_C(t) = \frac{q(t)}{C}, \qquad v_L(t) = L\frac{di(t)}{dt}, \qquad t \in \mathbb{R},$$

respectively, where q represents the charge of the capacitor. It follows that

$$u(t) = Ri(t) + \frac{q(t)}{C} + L\frac{di(t)}{dt}, \qquad t \in \mathbb{R}.$$

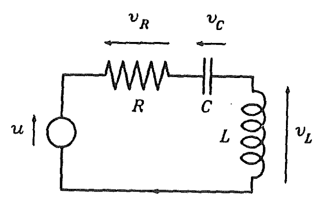


Figure 4.1. An RCL network.

Because $dq(t)/dt = i(t)$, we may differentiate to eliminate q. This results in

$$\frac{du(t)}{dt} = R\frac{di(t)}{dt} + \frac{i(t)}{C} + L\frac{d^2 i(t)}{dt^2}, \qquad t \in \mathbb{R}. \qquad (2)$$

(i) *Current i as output*. Suppose that the output is $y = i$. It immediately follows from (2) that

$$\frac{1}{LC}y(t) + \frac{R}{L}\frac{dy(t)}{dt} + \frac{d^2y(t)}{dt^2} - \frac{1}{L}\frac{du(t)}{dt} = 0, \qquad t \in \mathbb{R}.$$

This shows that the RCL network with the current as output is a second-order differential system.

(ii) *Voltage v_L as output*. If the output is $y = v_L = L\,di(t)/dt$, we need differentiate (2) once more. It follows that

$$\frac{d^2u(t)}{dt^2} = R\frac{d}{dt}\left(\frac{di(t)}{dt}\right) + \frac{1}{C}\left(\frac{di(t)}{dt}\right) + L\frac{d^2}{dt^2}\left(\frac{di(t)}{dt}\right), \qquad t \in \mathbb{R}.$$

By the substitution $L\,di/dt = y$ we find that the system is described by the differential equation

$$\frac{1}{LC}y(t) + \frac{R}{L}\frac{dy(t)}{dt} + \frac{d^2y(t)}{dt^2} - \frac{d^2u(t)}{dt^2} = 0, \qquad t \in \mathbb{R}.$$

This represents another second-order differential system.

(d) *Moving car*. The moving car of Example 3.2.13 is described by the first-order differential equation

$$M\frac{dv(t)}{dt} = cu(t) - Bv^2(t), \qquad t \in [t_o, \infty), \qquad (3)$$

with v representing the car speed and the input u the throttle position. Suppose that we take as output y the *distance* traveled by the car. Because

$$v(t) = \frac{dy(t)}{dt},$$

we may find the differential equation that describes the system with this output by the simple substitution $v = dy/dt$. This results in the differential equation

$$M\frac{d^2y(t)}{dt^2} + B\left(\frac{dy(t)}{dt}\right)^2 - cu(t) = 0, \qquad t \in \mathbb{R},$$

which represents a second-order differential system. ■

4.3 BASICS OF DIFFERENCE AND DIFFERENTIAL SYSTEMS

In this section we deal with a number of basic issues concerning difference and differential systems. First it is explained how by specifying *initial conditions* the output uniquely follows from the input, given the difference or differential equation. After studying non-anticipativeness, time-invariance, and linearity of difference and differential systems, we arrive at the systems that occupy our attention for most of the rest of this chapter, namely systems described by *constant coefficient linear difference* and *differential equations*. Finally, we consider *initially-at-rest* systems, which are the IOM systems that result by setting the initial conditions of the difference and differential equations equal to zero. This is a natural way of reducing IO difference and differential systems to IOM systems.

Solutions to Difference and Differential Equations

We devote some attention to the question how difference and differential equations as introduced in 4.2.1 define IO pairs.

The situation for difference systems is simple. Suppose that for each time n the equation

$$F[y(n), y(n+1), \cdots, y(n+N), u(n), u(n+1), \cdots, u(n+M), n] = 0$$

may uniquely be solved for $y(n+N)$ in the form

$$\begin{aligned} y(n+N) = G[&y(n), y(n+1), \cdots, y(n+N-1), \\ &u(n), u(n+1), \cdots, u(n+M), n], \qquad n \in \mathbb{T}. \end{aligned} \tag{1}$$

Take the time axis as $\mathbb{T} = \{n_o, n_o+1, n_o+2, \cdots\}$, and assume that the input u on this time axis is known. Then for $n = n_o$ the relation (1) yields $y(n_o+N)$, provided

$$y(n_o), y(n_o+1), \cdots, y(n_o+N-1)$$

are given. These numbers are called the *initial conditions*. Once $y(n_o+N)$ has been obtained, application of (1) for $n = n_o+1$ yields $y(n_o+N+1)$. Continuing in this way, the entire output $y(n)$ for $n \geq n_o+N$ may be constructed by what is called *successive substitution*.

Given the initial conditions and the input u, the output y is fully determined. Hence, by specifying the initial conditions the IO system becomes an input-output *mapping* system.

In the continuous-time case matters are more complicated. For the sake of the argument it is helpful to assume that the input u is M times continuously differentiable. Suppose that for each time t the relation

$$F\left[y(t), \frac{dy(t)}{dt}, \cdots, \frac{d^Ny(t)}{dt^N}, u(t), \frac{du(t)}{dt}, \cdots, \frac{d^Mu(t)}{dt^M}, t\right] = 0$$

may uniquely be solved for $d^Ny(t)/dt^N$ in the form

$$\frac{d^Ny(t)}{dt^N} = G\left[y(t), \frac{dy(t)}{dt}, \cdots, \frac{d^{N-1}y(t)}{dt^{N-1}}, u(t), \frac{du(t)}{dt}, \cdots, \frac{d^Mu(t)}{dt^M}, t\right]. \tag{2}$$

This relation shows that given the input u, at any time t the Nth derivative $d^Ny(t)/dt^N = y^{(N)}(t)$ of the output y is determined by $y(t), y^{(1)}(t), \cdots, y^{(N-1)}(t)$. Thus, given the input u and the N numbers $y(t_o), y^{(1)}(t_o), \cdots, y^{(N-1)}(t_o)$ at some initial time t_o we may compute, at least approximately, the quantities $y, y^{(1)}, \cdots, y^{(N-1)}$ at a short time h later, because

$$\begin{aligned} y(t_o+h) &= y(t_o) + \int_{t_o}^{t_o+h} y^{(1)}(\tau)\,d\tau \approx y(t_o) + y^{(1)}(t_o)h, \\ y^{(1)}(t_o+h) &= y^{(1)}(t_o) + \int_{t_o}^{t_o+h} y^{(2)}(\tau)\,d\tau \approx y^{(1)}(t_o) + y^{(2)}(t_o)h, \\ &\cdots \\ y^{(N-1)}(t_o+h) &= y^{(N-1)}(t_o) + \int_{t_o}^{t_o+h} y^{(N)}(\tau)\,d\tau \approx y^{(N-1)}(t_o) + y^{(N)}(t_o)h, \end{aligned}$$

where $y^{(N)}(t_o)$ in the latter expression is obtained from (2). By repeating this procedure at the time t_o+h we may find, at least approximately, y and its derivatives $y^{(1)}, \cdots, y^{(N-1)}$ at time t_o+2h, and continuing, at the times t_o+3h, t_o+4h, $\cdots$. It is plausible that if the time interval h is small enough we may obtain an arbitrarily accurate approximation to the solution of the differential equation for $t \geq t_o$.

The role of the *initial conditions* at time t_o is taken by the N numbers

$$y(t_o), y^{(1)}(t_o), \cdots, y^{(N-1)}(t_o).$$

It may be proved that under certain hypotheses on the function G, which in the cases we are interested in are usually satisfied, the solution of the differential equation

(2) for $t \geq t_o$ indeed is uniquely defined by the input together with the initial conditions. Like difference systems, differential systems are IO systems that become input-output *mapping* systems once the initial conditions have been specified.

4.3.1. Summary: Difference and differential IO and IOM systems.

A discrete-time system with time axis $\mathbb{T} = \{n_o, n_o + 1, \cdots\}$ and rule

$$F[y(n), y(n+1), \cdots, y(n+N), u(n), u(n+1), \cdots, u(n+M), n] = 0, \quad (3)$$

$n \in \mathbb{T}$, represents an IO system. Suppose that (3) may be written in the form

$$y(n+N) = G[y(n), y(n+1), \cdots, y(n+N-1), u(n), u(n+1), \cdots, u(n+M), n],$$

$n \in \mathbb{T}$. Then for given *initial conditions*

$$y(n_o), y(n_o+1), \cdots, y(n_o+N-1)$$

the system becomes an IOM system, which has a unique output y for any given input u.

A continuous-time system with time axis $\mathbb{T} = [t_o, \infty)$ and rule

$$F\left[y(t), \frac{dy(t)}{dt}, \cdots, \frac{d^N y(t)}{dt^N}, u(t), \frac{du(t)}{dt}, \cdots, \frac{d^M u(t)}{dt^M}, t\right] = 0 \quad (3')$$

$t \in \mathbb{T}$, represents an IO system. Suppose that (3′) may be written in the form

$$\frac{d^N y(t)}{dt^N} = G\left[y(t), \frac{dy(t)}{dt}, \cdots, \frac{d^{N-1} y(t)}{dt^{N-1}}, u(t), \frac{du(t)}{dt}, \cdots, \frac{d^M u(t)}{dt^M}, t\right],$$

$t \in \mathbb{T}$. Then for given *initial conditions*

$$y(t_o), y^{(1)}(t_o), \cdots, y^{(N-1)}(t_o)$$

the system becomes an IOM system, which has a unique output y for any given input u. ∎

If in the continuous-time case the input u is not M times continuously differentiable, its derivatives contain δ-functions and derivatives of δ-functions. As a result, the corresponding output y may also contain singular functions, and the differential equation is required to hold in the *distribution sense*. For linear differential systems, discussed in the next section, this is the usual situation.

Non-Anticipativity of Difference and Differential IOM Systems

The non-anticipativity of a difference IOM system with given initial conditions may easily be checked from its difference equation. Differential IOM systems are always non-anticipating.

4.3.2. Summary: Anticipativity of difference and differential IOM systems.

The difference system described by the difference equation

$$y(n+N) = G[y(n), y(n+1), \cdots, y(n+N-1), u(n), u(n+1), \cdots, u(n+M), n],$$

for $n \in \mathbb{T} = \{n_o, n_o+1, \cdots\}$, with given initial conditions $y(n_o)$, $y(n_o+1)$, $\cdots$, $y(n_o+N-1)$, is non-anticipating if $M \leq N$. If $M > N$ and the function G depends nontrivially on $u(n+N+1)$, $u(n+N+2)$, $\cdots$, $u(n+M)$, the system is anticipating.

The differential system described by t differential equation

$$\frac{dy^N(t)}{dt^N} = G\left[y(t), \frac{dy(t)}{dt}, \cdots, \frac{d^{N-1}y(t)}{dt^{N-1}}, u(t), \frac{du(t)}{dt}, \cdots, \frac{du^M(t)}{dt^M}, t\right]$$

for $t \in \mathbb{T} = [t_o, \infty)$, with given initial con ditions $y(t_o)$, $y^{(1)}(t_o)$, $\cdots$, $y^{(N-1)}(t_o)$, is non-anticipating. ∎

The proof is simple for the discrete-time case. The continuous-time result is plausible because the differential equation shows that the instantaneous changes in y and its derivatives at any time t depend on u and its derivatives at that same time and not on those at later times.

4.3.3. Examples: Anticipativeness.

(a) *Exponential smoother and RC network.* Both the exponential smoother of Example 4.2.3(a) and the RC network of 4.2.3(c) are non-anticipating.

(b) *Sliding window averager.* As we saw in Example 4.2.4(a), the discrete-time sliding window averager is described by the difference equation

$$y(n+M) = \frac{1}{N+M+1} \sum_{k=0}^{N+M} u(n+k), \qquad t \in \mathbb{T}.$$

It follows by 4.3.2 that the system is anticipating if and only if $N > 0$. This confirms what we found in 3.2.8(b). ∎

Time-Invariance of Difference and Differential Systems

We next discuss the time-invariance of difference and differential systems. Time-invariance may usually immediately be verified from the difference or differential equation by checking whether this equation only depends on time via u or y.

4.3.4. Summary: Time-invariance of difference and differential systems.

Consider the difference IO system described by the difference equation

$$F[y(n), y(n+1), \cdots, y(n+N), u(n), u(n+1), \cdots, u(n+M), n] = 0,$$

$n \in \mathbb{T}$, with $\mathbb{T} = \{n_o, n_o+1, \cdots\}$ or $\mathbb{T} = \mathbb{Z}$. Then if the function F does not depend on its last argument, that is,

$$F(y_0, y_1, \cdots, y_N, u_0, u_1, \cdots, u_M, n) = F(y_0, y_1, \cdots, y_N, u_0, u_1, \cdots, u_M, n')$$

for all $n, n' \in \mathbb{T}$ and for all $y_0, y_1, \cdots, y_N, u_0, u_1, \cdots, u_M$ in $\mathbb{C}$, the system is time-invariant on the time axis $\mathbb{T}$.

Consider the differential IO system described by the differential equation

$$F\left[y(t), \frac{dy(t)}{dt}, \cdots, \frac{d^N y(t)}{dt^N}, u(t), \frac{du(t)}{dt}, \cdots, \frac{du^M(t)}{dt^M}, t\right] = 0,$$

$t \in \mathbb{T}$, with $\mathbb{T} = [t_o, \infty)$ or $\mathbb{T} = \mathbb{R}$. Then if the function F does not depend on its last argument, that is,

$$F(y_0, y_1, \cdots, y_N, u_0, u_1, \cdots, u_M, t) = F(y_0, y_1, \cdots, y_N, u_0, u_1, \cdots, u_M, t')$$

for all $t, t' \in \mathbb{T}$ and for all $y_0, y_1, \cdots, y_N, u_0, u_1, \cdots, u_M$ in $\mathbb{C}$, the system is time-invariant on the time axis $\mathbb{T}$. ∎

Note that the condition that is given is *sufficient* for time-invariance, but not necessary. The trivial discrete- or continuous-time system described by $\alpha(t)y(t) - \alpha(t)u(t) = 0$, $t \in \mathbb{T}$, with α a nonzero time-dependent coefficient, for instance, does not satisfy the condition of 4.3.4. Nevertheless the system is time-invariant, because division by $\alpha(t)$ results in the equivalent equation $y(t) - u(t) = 0$, $t \in \mathbb{T}$, which by 4.3.4 represents a time-invariant system.

4.3.5. Proof. We only give the proof for the discrete-time case. That for the continuous-time case is similar. Suppose that (u, y) is an IO pair. Then

$$F[y(n), y(n+1), \cdots, y(n+N), u(n), u(n+1), \cdots, u(n+M), n] = 0,$$

$n \in \mathbb{T}$. Let θ be such that $n + \theta \in \mathbb{T}$ for all $n \in \mathbb{T}$. It follows that

$$F[y(n+\theta), y(n+\theta+1), \cdots, y(n+\theta+N), u(n+\theta), u(n+\theta+1), \cdots, u(n+\theta+M), n+\theta] = 0, \qquad n \in \mathbb{T}.$$

Because F does not depend on its last argument, we may also write

$$F[y(n+\theta), y(n+\theta+1), \cdots, y(n+\theta+N), u(n+\theta), u(n+\theta+1), \cdots, u(n+\theta+M), n] = 0, \qquad n \in \mathbb{T},$$

which shows that $(\sigma^\theta y, \sigma^\theta u)$ is an IO pair. Hence, the system is time-invariant. ∎

We consider some examples.

4.3.6. Examples: Time-invariant difference and differential systems.

(a) *Time-invariant systems.* The difference and differential systems we encountered so far, such as the exponential smoother of Example 4.2.3(a), the savings account of 4.2.3(b), the discrete-time sliding window averager of 4.2.4(a), the second-order smoother of 4.2.4(b), the RC network of 4.2.3(c), the moving car of 4.2.3(d) and 4.2.4(d), and the RCL network of 4.2.4(c) are all time-invariant.

(b) *RC network with time-dependent resistor.* For an example of a time-varying system suppose that the resistance of the resistor of the RC network of 1.2.7 is not constant, but a function $R(t)$, $t \in \mathbb{R}$, of time. Referring to Fig. 4.2 we have as in 1.2.7

$$\frac{dy(t)}{dt} = -\frac{1}{R(t)C}y(t) + \frac{1}{R(t)C}u(t), \qquad t \in \mathbb{R},$$

provided $R(t) > 0$ for all t. Because the coefficients of this differential equation vary with time, 4.3.4 is not satisfied. Indeed it may be verified that the system is time-invariant if and only if the resistance R is *constant*.

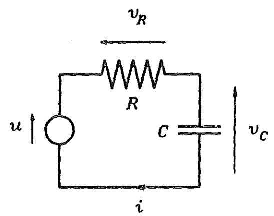


Figure 4.2. RC network.

(c) *RC network with time-dependent capacitor.* Next suppose that the resistance R is constant, but the capacitance $C(t)$, $t \in \mathbb{R}$, of the capacitor of the RC network varies with time. The equations

$$u(t) = v_R(t) + v_C(t), \tag{4}$$

and

$$v_R(t) = Ri(t) \tag{5}$$

remain valid, but the relation $i(t) = C dv_C(t)/dt$ need be replaced with $i(t) = dq(t)/dt$, where the charge q of the capacitor is given by $q(t) = C(t)v_C(t)$. It follows that $i(t) = \dot{C}(t)v_C(t) + C(t)dv_C(t)/dt$, where $\dot{C}$ is the derivative of C with respect to time. As a result, we obtain from (4) and (5) that

$$u(t) = R\left[\dot{C}(t)v_C(t) + C(t)\frac{dv_C(t)}{dt}\right] + v_C(t).$$

The substitution $v_C = y$ and rearrangement yield the differential equation

$$\frac{dy(t)}{dt} + \left(\frac{1}{RC(t)} + \frac{\dot{C}(t)}{C(t)}\right)y(t) = \frac{1}{RC(t)}u(t), \qquad t \in \mathbb{R},$$

provided $C(t) > 0$ for all t. This differential equation has time-dependent coefficients and represents a time-varying system. Note that the differential equation does *not* simply follow by replacing the constant C in the differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}u(t), \qquad t \in \mathbb{R}.$$

with the time-dependent parameter $C(t)$. ■

Linearity of Difference and Differential Systems

Also the linearity of a difference or differential system may often immediately be established by inspection of the difference or differential equation. If this equation is linear, so is the system.

4.3.7. Linearity of difference and differential systems.

The difference system described by the linear difference equation

$$\begin{aligned} q_0(n)y(n) + q_1(n)y(n+1) + \cdots \\ + q_N(n)y(n+N) \\ = p_0(n)u(n) + p_1(n)u(n+1) + \cdots \\ + p_M(n)u(n+M), \end{aligned}$$

$n \in \mathbb{T}$, with the q_k and p_k real or complex possibly time-dependent coefficients, is linear.

The differential system described by the linear differential equation

$$\begin{aligned} q_0(t)y(t) + q_1(t)\frac{dy(t)}{dt} + \cdots \\ + q_N(t)\frac{d^Ny(t)}{dt^N} \\ = p_0(t)u(t) + p_1(t)\frac{du(t)}{dt} + \cdots \\ + p_M(t)\frac{d^Mu(t)}{dt^M}, \end{aligned}$$

$t \in \mathbb{T}$, with the q_k and p_k real or complex possibly time-dependent coefficients, is linear. ■

Again, 4.3.7 gives a *sufficient* condition for linearity. An example of a trivial diff ence or differential equation that does not satisfy the condition but nevertheless r resents a linear system is $[1 + |y(t)|^2]y(t) = [1 + |y(t)|^2]u(t)$, $t \in \mathbb{T}$.

4.3.8. Proof of 4.3.7. This time we present the proof for the continuous-tir case. That for the discrete-time case is analogous. To prove linearity we need shc that if (u, y) and (u', y') are two IO pairs, any linear combination of these pairs also an IO pair. Because (u, y) and (u', y') are IO pairs we have

$$\begin{aligned} q_0(t)y(t) + q_1(t)\frac{dy(t)}{dt} + \cdots + q_N(t)\frac{d^Ny(t)}{dt^N} \\ = p_0(t)u(t) + p_1(t)\frac{du(t)}{dt} + \cdots + p_M(t)\frac{d^Mu(t)}{dt^M}, \end{aligned} \qquad (6)$$

for $t \in \mathbb{T}$, and

$$\begin{aligned} q_0(t)y'(t) + q_1(t)\frac{dy'(t)}{dt} + \cdots + q_N(t)\frac{d^Ny'(t)}{dt^N} \\ = p_0(t)u'(t) + p_1(t)\frac{du'(t)}{dt} + \cdots + p_M(t)\frac{d^Mu'(t)}{dt^M}, \end{aligned} \qquad (7)$$

for $t \in \mathbb{T}$. Multiplying (6) by any complex scalar α, (7) by α' and adding the results it follows that

$$\begin{aligned} q_0(t)[\alpha y(t) + \alpha' y'(t)] + q_1(t)\frac{d[\alpha y(t) + \alpha' y'(t)]}{dt} + \cdots \\ + q_N(t)\frac{d^N[\alpha y(t) + \alpha' y'(t)]}{dt^N} \\ = p_0(t)[\alpha u(t) + \alpha' u'(t)] + p_1(t)\frac{d[\alpha u(t) + \alpha' u'(t)]}{dt} + \cdots \\ + p_M(t)\frac{d^M[\alpha u(t) + \alpha' u'(t)]}{dt^M}, \end{aligned}$$

for $t \in \mathbb{T}$, which proves that $(\alpha u + \alpha' u', \alpha y + \alpha' y')$ is an IO pair. Hence, the system is linear. ■

We look at some examples.

4.3.9. Examples: Linearity.

(a) *Linear difference and differential systems.* Most difference and differential systems we have seen so far are linear, notably the exponential smoother of Example

4.2.3(a), the savings account of 4.2.3(b), the discrete-time sliding window averager of 4.2.4(a), the second-order smoother of 4.2.4(b), the RC network of 4.2.3(c), and the RCL network of 4.2.4(c). Also, the time-varying versions of the RC network of 4.3.6(b) and 4.3.6(c) are linear systems.

(b) *Moving car.* The moving car of Example 4.2.3(d) is described by the differential equation

$$M\frac{dv(t)}{dt} = cu(t) - Bv^2(t), \qquad t \geq t_0.$$

The differential equation is not linear. Indeed, the system is not linear. For instance, if u_o is a constant input, $v_o = \sqrt{cu_o/B}$ is a corresponding output, but $(\alpha u_o, \alpha v_o)$ is not an IO pair unless $\alpha = 0$, $\alpha = 1$ or $u_o = 0$.

Also when we take the position of the car as output, as in Example 4.2.4(d), the system is nonlinear. ■

Linear Time-Invariant Difference and Differential Systems

An important and useful class of difference and differential systems consists of those difference and differential systems that are both linear and time-invariant. By combining 4.3.4 and 4.3.7 we obtain the following result.

4.3.10. Summary: Linear time-invariant difference and differential systems.

The difference system described by the linear constant coefficient difference equation

$$\begin{aligned} q_0 y(n) + q_1 y(n+1) + \cdots & \\ + q_N y(n+N) & \\ = p_0 u(n) + p_1 u(n+1) + \cdots & \\ + p_M u(n+M), & \qquad (8) \end{aligned}$$

$n \in \mathbb{T}$, with the q_k and p_k real or complex coefficients, is linear and time-invariant.

The differential system described by the linear constant coefficient differential equation

$$\begin{aligned} q_0 y(t) + q_1\frac{dy(t)}{dt} + \cdots + q_N\frac{d^N y(t)}{dt^N} & \\ = p_0 u(t) + p_1\frac{du(t)}{dt} + \cdots + p_M\frac{d^M u(t)}{dt^M}, & \qquad (8') \end{aligned}$$

$t \in \mathbb{T}$, with the q_k and p_k real or complex coefficients, is linear and time-invariant. ■

We next introduce a useful notation. Let Q and P be the polynomials

$$Q(\lambda) = q_0 + q_1\lambda + \cdots + q_N\lambda^N,$$
$$P(\lambda) = p_0 + p_1\lambda + \cdots + p_M\lambda^M.$$

We may then write the difference equation (8) and the differential equation (8') in the more compact forms

$$Q(\sigma)y = P(\sigma)u,$$

or

$$Q(D)y = P(D)u,$$

respectively. Here σ is the *back shift operator* defined by

$$\sigma y(n) = y(n+1), \qquad n \in \mathbb{T},$$

and D the *differential operator* defined by

$$Dy(t) = \frac{dy(t)}{dt}, \qquad t \in \mathbb{T}.$$

4.3.11. Examples: Linear time-invariant difference and differential systems. Many of the difference and differential systems we have seen are both linear and time-invariant. To this category belong the exponential smoother of 4.2.3(a), the savings account of 4.2.3(b), the discrete-time sliding window averager of 4.2.4(a), the second-order smoother of 4.2.4(b), the RC circuit of 4.2.3(c), and the RCL network of 4.2.4(c). The moving car of 4.2.3(d) and 4.2.4(d) is time-invariant but nonlinear. The RC network with time-varying resistor of 4.3.4(b) or time-varying capacitor of 4.3.4(c) is linear but time-varying. ■

The Initially-At-Rest System

Suppose that the RCL network of 4.2.4(c) at some given time contains electrical or magnetic energy. If the input voltage is kept at zero, the energy dissipates, and eventually reduces to zero. Such an energy-free circuit is said to be *at rest*. A system that is at rest has the property that its response to a zero input is also zero. Generally, if the system is at rest it has a unique response to any given (nonzero) input. The input-output mapping system thus defined is called the *initially-at-rest* system.

We now present a precise definition of initially-at-rest difference and differential systems. It is simplest to consider only systems defined on the infinite time axis $\mathbb{Z}$ or $\mathbb{R}$. First we need establish conditions such that the zero input and zero output actually form an IO pair. Consider the difference system described by the difference equation

$$\begin{aligned} y(n+N) = G[&y(n), y(n+1), \cdots, y(n+N-1), \\ &u(n), u(n+1), \cdots, u(n+M), n], \qquad n \in \mathbb{Z}, \end{aligned} \qquad (9)$$

or the differential system described by the differential equation

$$\frac{dy^N(t)}{dt^N} = G[y(t), \frac{dy(t)}{dt}, \cdots, \frac{dy^{(N-1)}(t)}{dt^{N-1}}, u(t), \frac{du(t)}{dt}, \cdots, \frac{du^M(t)}{dt^M}, t], \qquad t \in \mathbb{R}. \tag{10}$$

Then if the function G in (9) or (10) has the property

$$G(0, 0, \cdots, 0, t) = 0 \qquad \text{for all} \quad t \in \mathbb{Z} \text{ or } \mathbb{R},$$

the difference system (9) or the differential system (10) has (0, 0) as an IO pair.

We next look for conditions such that the zero input $u = 0$ has the zero output $y = 0$ as its *unique* response. First, consider the difference system. Let n_o be some instant of time on the discrete time axis $\mathbb{Z}$, and suppose that

$$y(n_o - N) = y(n_o - N + 1) = \cdots = y(n_o - 1) = 0,$$

and

$$u(n_o - N) = u(n_o - N + 1) = \cdots = u(n_o - 1) = 0.$$

It easily follows by successive substitution that the unique response to the input $u(n) = 0$ for $n = n_o, n_o + 1, \cdots$, is $y(n) = 0$ for $n = n_o, n_o + 1, \cdots$. Moreover, to any *nonzero* input u, restricted to $n_o, n_o + 1, \cdots$, there corresponds a unique response $y_{n_o}(n)$, $n = n_o, n_o + 1, \cdots$.

Now, let $n_o \to -\infty$. The limit of y_{n_o}, if it exists, is denoted as y. We take the limit *pointwise*, that is,

$$y(n) = \lim_{n_o \to -\infty} y_{n_o}(n)$$

for any fixed n. The IOM system consisting of all IO pairs (u, y) thus defined is called the *initially-at-rest* system. Note that (0, 0) is an IO pair. Also note that any right one-sided input maps into a right one-sided output.

Before going on to the continuous-time case we present an example.

4.3.12. Example: Initially-at-rest exponential smoother. The exponential smoother is described by the first-order difference equation

$$y(n + 1) - ay(n) = (1 - a)u(n + 1), \qquad n \in \mathbb{Z}.$$

Because $N = 1$, the conditions that ensure the system to be at rest at time n_o are

$$y(n_o - 1) = 0, \qquad u(n_o - 1) = 0.$$

The latter condition happens to be superfluous for this particular example beca the difference equation has no term with $u(n)$ on the right hand side. By 3.4.2(for $y(n_o - 1) = 0$ the solution to the difference equation is

$$y(n) = (1 - a) \sum_{m=n_o}^{n} a^{n-m} u(m), \qquad n = n_o, n_o + 1, n_o + 2, \cdots.$$

Letting n_o approach $-\infty$ we obtain

$$y(n) = (1 - a) \sum_{m=-\infty}^{n} a^{n-m} u(m), \qquad n \in \mathbb{Z},$$

which is defined for all u for which the infinite sum exists. This is the IO map of th initially-at-rest exponential smoother. In fact, this is a *convolution map* of the forn

$$y = h * u,$$

with

$$h(n) = (1 - a)a^n \mathbb{1}(n), \qquad n \in \mathbb{Z}.$$

The initially-at-rest exponential smoother thus is a *convolution system* with impulse response h. ∎

We next discuss initially-at-rest differential systems. Suppose that for the differential system defined by (10) we have

$$y(t_o^-) = y^{(1)}(t_o^-) = \cdots = y^{(N-1)}(t_o^-) = 0, \tag{11}$$

and

$$u(t_o^-) = u^{(1)}(t_o^-) = \cdots = u^{(M-1)}(t_o^-) = 0, \tag{12}$$

for some $t_o \in \mathbb{R}$. Then the unique response to the input $u(t) = 0$ for $t \geq t_o$ is $y(t) = 0$, $t \geq t_o$. Note that we need condition (12) on the derivatives of u at time t_o^- (i.e., just before time t_o) to prevent any delta functions or derivatives of delta functions at time t_o to enter into the right-hand side of (10). For any *nonzero* input u, restricted to $[t_o, \infty)$, the initial conditions (11) and (12) define a unique response $y_{t_o}(t)$, $t \geq t_o$. Letting $t_o \to -\infty$ we denote the limit of y_{t_o}, if it exists, as y. We say that y is the response to the input u of the *initially-at-rest* differential system derived from (10). As in the discrete-time case, the initially-at-rest system has (0, 0) as an IO pair. Right one-sided inputs results in right one-sided outputs.

Both in the discrete- and continuous-time case inputs for which the response does not exist as the initial time approaches $-\infty$ are excluded from the input set.

4.3.13. Example: Initially-at-rest RC network. The RC network is a first-order differential system described by the differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}u(t), \qquad t \in \mathbb{R}.$$

The system is at rest at time t_o if $y(t_o^-) = 0$. In 3.4.2(b) we saw that the output of the RC network may be expressed as

$$y(t) = e^{-(t-t_o)/RC}y(t_o) + \frac{1}{RC}\int_{t_o}^{t} e^{-(t-\tau)/RC}u(\tau)\,d\tau, \qquad t \geq t_o.$$

Replacing t_o with t_o^- and setting $y(t_o^-) = 0$ we obtain

$$y(t) = \frac{1}{RC}\int_{t_o^-}^{t} e^{-(t-\tau)/RC}u(\tau)\,d\tau, \qquad t \geq t_o^-.$$

By letting t_o approach $-\infty$ we see that the IO map of the initially-at-rest RC network is given by

$$y(t) = \frac{1}{RC}\int_{-\infty}^{t} e^{-(t-\tau)/RC}u(\tau)\,d\tau, \qquad t \in \mathbb{R},$$

for those inputs u for which the integral exists. Inspection shows that the initially-at-rest RC network is a continuous-time convolution system

$$y = h * u$$

with impulse response

$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}\mathbb{1}(t), \qquad t \in \mathbb{R}.$$ ■

The examples show that both the initially-at-rest exponential smoother and the initially-at-rest RC network are convolution systems. In fact, *all* initially-at-rest difference and differential systems defined by constant coefficient difference and differential systems are convolution systems. The reason is that initially-at-rest difference and differential systems are (1) linear, (2) time-invariant, and (3) IOM systems. Both (1) and (2) may easily be proved. By 3.4.1, linear time-invariant IOM systems on the time axis $\mathbb{Z}$ or $\mathbb{R}$ are convolution systems.

4.3.14. Summary: Initially-at-rest linear time-invariant difference and differential systems are convolution systems. The initially-at-rest system defined by the constant coefficient linear difference equation

$$Q(\sigma)y = P(\sigma)u$$

on the time axis $\mathbb{Z}$, or the differential equation

$$Q(D)y = P(D)u$$

on the time axis $\mathbb{R}$, is a linear time-invariant IOM system, and, hence, a convolution system. ■

We conclude by reviewing the main results of this section for sampled systems.

4.3.15. Sampled difference systems. Sampled difference systems are defined by difference equations of the form

$$F[y(t), y(t+T), \cdots, y(t+NT), u(t), u(t+T), \cdots, u(t+MT), t] = 0$$

for $t \in \mathbb{T}$, where $\mathbb{T} = \{t_o, t_o + T, t_o + 2T, \cdots\}$ with $t_o \in \mathbb{Z}(T)$, or $\mathbb{T} = \mathbb{Z}(T)$. We assume that for each $t \in \mathbb{T}$ this equation may uniquely be solved for $y(t + NT)$ in the form

$$\begin{aligned} y(t+NT) = G[&y(t), y(t+T), \cdots, y(t+(N-1)T), \\ &u(t), u(t+T), \cdots, u(t+MT), t], \qquad t \in \mathbb{T}. \end{aligned}$$

N is the *order* of the system. The solution from time t_o on is fully determined by the initial conditions

$$y(t_o), y(t_o + T), \cdots, y(t_o + (N-1)T)$$

together with the input u from time t_o on. The resulting IOM system is *anticipating* if and only if both $M > N$ and G depends nontrivially on $u(t + (N+1)T)$, $u(t + (N+2)T), \cdots, u(t + MT)$.

The sampled difference system is *time-invariant* if the function F does not depend on its last argument, and *linear* if the function F is linear in its first $N + M + 2$ arguments.

If the difference equation is the constant coefficient linear difference equation

$$\begin{aligned} q_0 y(t) + q_1 y(t+T) + \cdots + q_N y(t+NT) \\ = p_0 u(t) + p_1 u(t+T) + \cdots + p_M u(t+MT), \qquad t \in \mathbb{T}, \end{aligned}$$

with the q_k and p_k real or complex coefficients, the system is both linear and time-invariant. Defining the polynomials

$$Q(\lambda) = q_0 + q_1\lambda + \cdots + q_N\lambda^N, \qquad P(\lambda) = p_0 + p_1\lambda + \cdots + p_M\lambda^M,$$

the difference equation may be written in the compact form

$$Q(\sigma^T)y = P(\sigma^T)u,$$

with σ the back shift operator.

Initially-at-rest sampled difference systems are defined analogously to initially-at-rest difference systems. Initially-at-rest sampled difference systems defined by constant coefficient linear difference equations are linear time-invariant and, hence, are sampled convolution systems. ∎

4.4 RESPONSE OF LINEAR TIME-INVARIANT DIFFERENCE AND DIFFERENTIAL SYSTEMS

In this section we study linear time-invariant difference and differential systems described by constant coefficient linear difference and differential equations. The difference equations are of the form

$$Q(\sigma)y = P(\sigma)u,$$

while the differential equations look like

$$Q(D)y = P(D)u.$$

Q and P are the polynomials

$$Q(\lambda) = q_0 + q_1\lambda + \cdots + q_N\lambda^N, \tag{1a}$$

$$P(\lambda) = p_0 + p_1\lambda + \cdots + p_M\lambda^M. \tag{1b}$$

Without loss of generality we assume in the rest of this chapter that the *leading coefficients* q_N and p_M are both nonzero.

The time axis $\mathbb{T}$ is taken to be *right semi-infinite* or *infinite*. In the discrete-time case the appropriate time axes are

$$\text{Semi-infinite:}\quad \mathbb{T}_+ = \{n_o, n_o + 1, n_o + 2, \cdots\},$$
$$\text{Infinite:}\quad \mathbb{T}_\infty = \mathbb{Z},$$

with $n_o \in \mathbb{Z}$. In the continuous-time case we use the time axes

$$\text{Semi-infinite:}\quad \mathbb{T}_+ = [t_o, \infty),$$
$$\text{Infinite:}\quad \mathbb{T}_\infty = \mathbb{R},$$

with $t_o \in \mathbb{R}$. Because in the continuous-time case we wish to allow the input to hav δ-functions at time t_o, the semi-infinite time axis $\mathbb{T}_+$ is understood to include th time t_o^-. This means that the time axis really is the interval $\mathbb{T}_+ = (t_o - \epsilon, \infty)$ wit $\epsilon \downarrow 0$.

Difference and differential systems are IO systems but not IOM systems. In th sequel we study how to find the set of *all* possible outputs y corresponding to a giver input u. We also consider difference and differential systems with *given* initial conditions, so that the IO system becomes an IOM system, and see how we may determine the unique output y corresponding to any given input u. In Section 4.5 we investigate in particular initially-at-rest constant coefficient linear difference and differential systems.

The plan of the present section is as follows:

(i) First we study the *homogeneous* solution of linear constant coefficient difference and differential equations. This gives us *all* input-output pairs of the form $(0, y)$, that is, the set of all *zero-input responses* of the system.
(ii) Next we introduce the idea of a *particular solution* of the difference or differential equation for a given input u. A particular solution y_{part} is *any* solution of the difference or differential equation such that (u, y_{part}) forms an IO pair.
(iii) Given a particular solution y_{part} corresponding to some input u, and given the set of homogeneous solutions we may determine the *general* solution of the difference or differential equation. The general solution defines the set of all input-output pairs (u, y) for a given input u.
(iv) Given the set of all IO pairs for a given input u we may use initial conditions to select a special solution.

In Section 4.5, where we consider initially-at-rest linear time-invariant difference and differential systems, it will be seen how particular solutions of the difference or differential equation may be constructed if the impulse response of the intially-at-rest system is known.

Solution of the Homogeneous Equation

The *homogeneous* equation corresponding to the difference equation $Q(\sigma)y = P(\sigma)u$ or the differential equation $Q(D)y = P(D)u$ is obtained by setting the right-hand side equal to zero, resulting in

$$Q(\sigma)y = 0$$

or

$$Q(D)y = 0,$$

respectively. The solution of the homogeneous equation is entirely determined by the roots of the polynomial Q. These roots are called the *characteristic roots* of the difference or differential system.

The following facts are well-known from the elementary theory of difference and differential equations.

4.4.1. Summary: Solution of the homogeneous equation.

(a) On the time axis $\mathbb{T}_+ = \{n_o, n_o + 1, n_o + 2, \cdots\}$, with $n_o \in \mathbb{Z}$, the homogeneous difference equation $Q(\sigma)y = 0$ has N linearly independent solutions $y_1, y_2, \cdots, y_N$, called *basis solutions*.

On the time axis $\mathbb{T}_\infty = \mathbb{Z}$ the homogeneous equation has N_o basis solutions $y_1, y_2, \cdots, y_{N_o}$, where N_o is the number of *nonzero* roots of Q.

(b) Any solution of the homogeneous difference equation is a linear combination of the basis solutions.

(c) One way of selecting basis solutions is to determine corresponding to each *nonzero* root λ of multiplicity m of the polynomial Q the m basis solutions given by

$$n^i\lambda^n, \qquad n \in \mathbb{T}_+ \text{ or } \mathbb{T}_\infty,$$

for $i = 0, 1, \cdots, m - 1$. On the time axis $\mathbb{T}_+$ the equation has the m_o *additional* basis solutions

$$\Delta(n - n_o - i), \qquad n \in \mathbb{T}_+,$$

for $i = 0, 1, \cdots, m_o - 1$, where $m_o = N - N_o$ is the number of *zero* roots of Q.

(a′) Both on the time axis $\mathbb{T}_+ = [t_o, \infty)$, with $t_o \in \mathbb{R}$, and $\mathbb{T}_\infty = \mathbb{R}$ the homogeneous differential equation $Q(D)y = 0$ has N linearly independent solutions $y_1, y_2, \cdots, y_N$, called *basis solutions*.

(b′) Any solution of the homogeneous differential equation is a linear combination of the basis solutions.

(c′) One way of selecting basis solutions is to determine corresponding to each root λ of multiplicity m of the polynomial Q the m basis solutions given by

$$t^i e^{\lambda t}, \qquad t \in \mathbb{T}_+ \text{ or } \mathbb{T}_\infty,$$

for $i = 0, 1, \cdots, m - 1$. ∎

The basis solutions $y_1, y_2, \cdots, y_N$ or $y_1, y_2, \cdots, y_{N_o}$ are said to be linearly independent if no nontrivial linear combination of the basis solutions is identical to zero. In (c), Δ is the pulse of 2.2.7(a), defined by

$$\Delta(n) = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{otherwise}, \end{cases} \qquad n \in \mathbb{Z}.$$

The main result of 4.4.1 is that any solution of the homogeneous difference or differential equation is a suitable linear combination of basis solutions. The basis solutions are not unique, but according to (c) and (c′) a convenient set of basis solutions may be found from the characteristic roots (i.e., the roots of the polynomial Q). For each root we find as many basis solutions as its multiplicity.

In the discrete-time case, zero characteristic roots play a special role. If the time axis is $\mathbb{T}_+ = \{n_o, n_o + 1, n_o + 2, \cdots\}$ and Q has m_o zero roots, we obtain m_o basis solutions consisting of shifted unit pulses, as shown in Fig. 4.3. If the time axis is $\mathbb{T}_\infty = \mathbb{Z}$, these basis solutions are missing (because they move "out of view" towards $-\infty$).

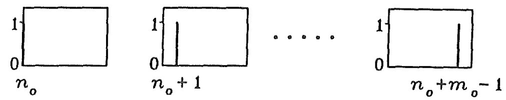


Figure 4.3. Discrete-time basis functions corresponding to a zero characteristic root of multiplicity m_o.

The result of 4.4.1. allows us to construct the set $\mathcal{Y}_0$ of all *zero-input* responses of the difference or differential system. It is given by

$$\mathcal{Y}_0 = \left\{y \mid y = \sum_i \alpha_i y_i,\ \alpha_i \in \mathbb{C} \text{ for each } i\right\},$$

with the y_i the basis solutions of the homogeneous difference or differential equation.

We consider some examples.

4.4.2. Examples: Solutions of homogeneous equations.

(a) *Exponential smoother*. The exponential smoother is described by the difference equation

$$y(n + 1) - ay(n) = (1 - a)u(n + 1), \qquad n \in \mathbb{T},$$

where $\mathbb{T}$ is either $\mathbb{T}_+ = \{n_o, n_o + 1, \cdots\}$ or $\mathbb{T}_\infty = \mathbb{Z}$. The polynomial Q is given by $Q(\lambda) = \lambda - a$, which has the single root $\lambda_1 = a$.

If $a \neq 0$ all solutions of the homogeneous equation are of the form

$$y_{\text{hom}}(n) = \alpha a^n, \qquad n \in \mathbb{T},$$

with α an arbitrary constant, possibly complex.

If $a = 0$ and the time axis is $\mathbb{T}_\infty$, the homogeneous equation is $y(n + 1) = 0$, $n \in \mathbb{T}_\infty$, which has the unique trivial solution

$$y_{\text{hom}}(n) = 0, \qquad t \in \mathbb{T}_\infty.$$

If $a = 0$ and the time axis is $\mathbb{T}_+$, all solutions of the homogeneous equation $y(n + 1) = 0$, $n \in \mathbb{T}_+$, are of the form

$$y_{\text{hom}}(n) = \alpha\Delta(n - n_o), \qquad n \in \mathbb{T}_+,$$

with α an arbitrary constant, possibly complex.

(b) *RCL network*. The RCL network of Example 4.2.4.(c) is both in case (i) and case (ii) described by a differential equation of the form $Q(D)y = P(D)u$, where Q is given by

$$Q(\lambda) = \frac{1}{LC} + \frac{R}{L}\lambda + \lambda^2,$$

while P depends on the choice of the output.

If $R^2 \neq 4L/C$, the system has two different characteristic roots, given by

$$\lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}.$$

All solutions of the homogeneous equation hence are of the form

$$y_{\text{hom}}(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t}, \qquad t \in \mathbb{T},$$

with α_1 and α_2 arbitrary constants, both on the time axis $\mathbb{T} = \mathbb{T}_+ = [t_o,\infty)$ and on $\mathbb{T} = \mathbb{T}_\infty = \mathbb{R}$.

If $R^2 > 4L/C$, the roots λ_1 and λ_2 are both real and y_{hom} consists of a linear combination of two real-valued decaying exponentials.

If $R^2 < 4L/C$, the roots form a complex conjugate pair, and the exponentials are complex-valued. In Example 4.4.4 we see that in this case the exponentials may be replaced by two real-valued decaying harmonics.

If $R^2 = 4L/C$, the polynomials Q has a root $-R/2L = -1/\sqrt{LC}$ of multiplicity 2. As a result, the homogeneous solution is of the form

$$y_{\text{hom}}(t) = \alpha_1 e^{-t/\sqrt{LC}} + \alpha_2 t e^{-t/\sqrt{LC}}, \qquad t \in \mathbb{T}.$$

When this solution applies the system is said to be *critically damped*. ∎

When a characteristic root λ of a difference or differential equation is real, the corresponding basis solution is real. Even if the polynomial Q has real coefficients, it may easily have complex roots, which make the corresponding basis solutions also complex. Often we are interested in real solutions only. If the polynomial Q has real coefficients (which is the usual situation), its complex roots always occur in complex conjugate pairs. The corresponding basis solutions may be combined to construct pairs of *real* basis solutions as follows.

4.4.3. Summary: Real basis solutions.

If the homogeneous difference equation $Q(\sigma)y = 0$ has a pair of complex conjugate roots λ, $\bar\lambda$ of multiplicity m, the equation has the $2m$ corresponding *real* basis solutions

$$n^i\rho^n \cos(\psi n), \quad n^i\rho^n \sin(\psi n), \qquad n \in \mathbb{T},$$

for $i = 0, 1, \cdots, m-1$, where

$$\rho := |\lambda|, \qquad \psi := \arg(\lambda).$$

If the homogeneous differential equati- $Q(D)y = 0$ has a pair of complex conj- gate roots λ, $\bar\lambda$ of multiplicity m, the equ- tion has the $2m$ corresponding *real* ba- solutions

$$t^i e^{\sigma t} \cos(\omega t), \quad t^i e^{\sigma t} \sin(\omega t), \qquad t \in \mathbb{T},$$

for $i = 0, 1, \cdots, m-1$, where

$$\sigma := \text{Re}(\lambda), \qquad \omega := \text{Im}(\lambda).$$ ∎

The proof is left as an exercise.

4.4.4. Example: Real basis solutions for the RCL network. In Example 4.4.2(b) we saw that if $R^2 \neq 4L/C$, then the polynomial Q of the RCL network has the two roots

$$\lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}.$$

We define the *resonance frequency* ω_r and the *quality factor* q of the network as

$$\omega_r = \frac{1}{\sqrt{LC}}, \qquad q = \frac{\omega_r L}{R}.$$

Then the characteristic roots may be expressed as

$$\lambda_{1,2} = \frac{\omega_r}{2q}(-1 \pm \sqrt{1 - 4q^2}).$$

For $q \leq 1/2$ the characteristic roots are both real, but if $q > 1/2$ they form a complex conjugate pair λ, $\bar\lambda$ with real and imaginary parts

$$\sigma = \text{Re}(\lambda) = -\frac{\omega_r}{2q}, \qquad \omega = \text{Im}(\lambda) = \omega_r\sqrt{1 - 1/4q^2}.$$

From 4.4.3 all solutions of the homogeneous equation may be expressed as

$$y_{\text{hom}}(t) = \alpha_1 e^{\sigma t}\cos(\omega t) + \alpha_2 e^{\sigma t}\sin(\omega t), \qquad t \in \mathbb{T}, \tag{2}$$

with α_1 and α_2 arbitrary constants. All real solutions of the homogeneous equation follow by taking α_1 and α_2 real. If α_1 and α_2 are real, (2) may alternatively be written as

$$\begin{aligned} y_{\text{hom}}(t) &= \alpha e^{\sigma t}\cos(\omega t + \phi), \\ &= \alpha e^{-\omega_r t/2q}\cos(\omega_r t\sqrt{1 - 1/4q^2} + \phi), \qquad t \in \mathbb{T}, \end{aligned} \tag{3}$$

with α and ϕ arbitrary real constants. This solution represents a damped harmonic with frequency $\omega = \omega_r\sqrt{1 - 1/4q^2}$ and time constant $2q/\omega_r$. If the quality factor q is large, the frequency ω is close to the resonance frequency ω_r and the damping is small.

Figure 4.4 shows the real basis solutions of the RCL network we obtained in 4.4.2(b) and those of the present example.

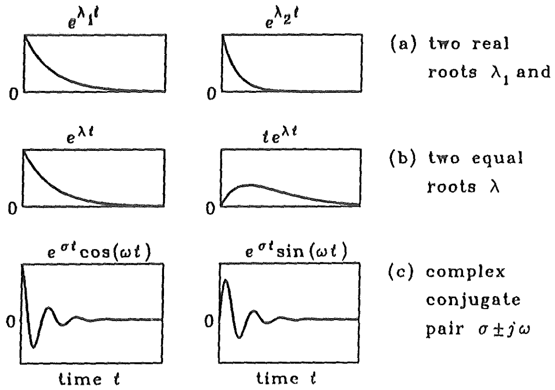


Figure 4.4. Real basis solutions of the RCL network. Top: two real roots. Middle: two equal roots. Bottom: complex conjugate pair of roots.

Exercise. Prove that if α_1 and α_2 are real, then (2) may be rewritten in the form (3) with α and ϕ real. ∎

Particular Solutions

A *particular solution* y_{part} of the difference equation

$$Q(\sigma)y = P(\sigma)u$$

or the differential equation

$$Q(D)y = P(D)u$$

corresponding to a given input u simply is *any* output y_{part} such that $Q(\sigma)y_{\text{part}} = P(\sigma)u$ or $Q(D)y_{\text{part}} = P(D)u$, respectively.

For some input signals u particular solutions are easy to guess. If the input is an exponential signal, for instance, there usually exists an exponential particular solution with the same parameter.

4.4.5. Example: Particular solutions for exponential inputs to the exponential smoother. The exponential smoother is described by the difference equation

$$y(n + 1) - ay(n) = (1 - a)u(n + 1), \qquad n \in \mathbb{T},$$

with $\mathbb{T}$ a right semi-infinite or infinite time axis. Suppose that the input is the exponential signal

$$u(n) = u_o z^n, \qquad n \in \mathbb{T},$$

with u_o and z given real or complex constants. We try to find a particular solution of a similar form $y(n) = y_o z^n$, $n \in \mathbb{T}$, with the constant y_o to be determined. Substitution of u and y into the difference equation yields

$$y_o z^{n+1} - ay_o z^n = (1 - a)u_o z^{n+1}, \qquad n \in \mathbb{T}.$$

Cancellation of the common factor z^n and solution for y_o results in

$$y_o = \frac{1 - a}{z - a}u_o z,$$

provided $z \neq a$. This shows that if $z \neq a$ for the given exponential input the difference equation has an exponential particular solution

$$y_{\text{part}}(n) = \frac{1 - a}{z - a}u_o z^{n+1}, \qquad n \in \mathbb{T}.$$

Exercise. Show that if $z = a$, the difference equation has the particular solution

$$y_{\text{part}}(n) = (1 - a)nu_o a^n, \qquad n \in \mathbb{T}.$$ ∎

General Solution of Difference and Differential Equations

We finally study the *general solution* of constant coefficient linear difference and differential equations. The general solution is a form in which every solution may be written.

4.4.6. Summary: General solution of the nonhomogeneous equation. Every solution of the constant coefficient linear difference equation $Q(\sigma)y = P(\sigma)u$ or the differential equation $Q(D)y = P(D)u$ for a given input u may be written as

$$y = y_{\text{part}} + y_{\text{hom}},$$

where y_{part} is a particular solution of the nonhomogeneous difference or the differential equation corresponding to the input u, and y_{hom} is a suitable linear combination

$$y_{\text{hom}} = \sum_i \alpha_i y_i, \qquad \alpha_i \in \mathbb{C} \text{ for each } i,$$

of the basis solutions y_i of the homogeneous equation. ∎

In other words, if we happen to know some solution y_{part} of the nonhomogeneous difference or differential equation (on the time axis of interest), any other solution may be obtained by adding a suitable solution of the homogeneous equation. It follows that the set of solutions $\mathcal{Y}_u$ of the difference or differential equations corresponding to the input u is given by

$$\mathcal{Y}_u = \left\{ y \mid y = y_{\text{part}} + \sum_i \alpha_i y_i, \qquad \alpha_i \in \mathbb{C} \text{ for each } i \right\}.$$

The arbitrary constants α_i occurring in the general solution may be used to satisfy N initial or boundary conditions, such as

$$y(0) = y_0, \qquad y(1) = y_1, \qquad \cdots, \qquad y(N-1) = y_{N-1},$$

in the discrete-time case, or

$$y(0) = y_0, \qquad y^{(1)}(0) = y_1, \qquad \cdots, \qquad y^{(N-1)}(0) = y_{N-1},$$

in the continuous-time case, where $y_0, y_1, \cdots, y_{N-1}$, are N given real or complex numbers.

4.4.7. Example: General solution and initial conditions.
(a) *Exponential smoother*. The exponential smoother is described by the difference equation

$$y(n+1) = ay(n) + (1-a)u(n+1), \qquad n = 0, 1, 2, \cdots,$$

where we take the time axis as $\mathbb{T} = \mathbb{Z}_+$. Suppose that the input is constant and given by

$$u(n) = 1, \qquad n = 0, 1, 2, \cdots.$$

We conjecture that there exists a constant particular solution $y(n) = y_o$, $n = 0, 2, \cdots$. Substitution into the difference equation results in

$$y_o = ay_o + (1-a), \qquad n = 0, 1, 2, \cdots,$$

which is satisfied for $y_o = 1$.

In Example 4.4.2(a) we found that for $a \neq 0$ the homogeneous equation ha the single basis solution $y_1(n) = a^n$, $n = 0, 1, 2, \cdots$. It follows from 4.4.6 th corresponding to the given input the difference equation has the general solution

$$y(n) = 1 + \alpha a^n, \qquad n = 0, 1, 2, \cdots, \qquad (4$$

with α an arbitrary constant.

The arbitrary constant α may be determined if an initial condition is given. Suppose that $y(0) = 0$. Substitution of $n = 0$ into (4) then yields

$$0 = 1 + \alpha,$$

so that $\alpha = -1$. It follows from (4) that the response of the system to the constant input $u(n) = 1$, $n = 0, 1, 2, \cdots$, with the initial condition $y(0) = 0$ is

$$y(n) = 1 - a^n, \qquad n = 0, 1, 2, \cdots.$$

(b) *RCL network*. As a second, more complicated example we consider the RCL network of Example 4.2.4(c). If we take the current i through the network as the output y, the network is described by the differential equation

$$y^{(2)}(t) + \frac{R}{L}y^{(1)}(t) + \frac{1}{LC}y(t) = \frac{1}{L}u^{(1)}(t), \qquad t \in \mathbb{T}.$$

Suppose that the input is constant and given by

$$u(t) = 1 \qquad \text{for } t \geq 0^-.$$

It is easy to verify that correspondingly the differential equation has the particular solution

$$y_{\text{part}}(t) = 0, \qquad t \geq 0^-.$$

In Example 4.4.2(b) we found that if $R^2 \neq 4L/C$ the homogeneous equation has the basis solutions $y_i(t) = \exp(\lambda_i t)$, $t \in \mathbb{T}$, $i = 1, 2$, where λ_1 and λ_2 are the characteristic roots. Thus, corresponding to the given constant input the general solution of the differential equation is

$$y(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t}, \qquad t \geq 0^-. \tag{5}$$

Suppose now that the initial conditions of the RCL network (see Fig. 4.1) are specified by the requirement that at time zero both the charge q of the capacitor and the flux ϕ contained in the inductor are zero. We determine the resulting initial conditions on y. Since the flux through the inductor equals $\phi = Li = Ly$ it follows immediately from $\phi(0) = 0$ that $y(0) = 0$, which provides us with the first initial condition. Substitution of $t = 0$ into (5) yields

$$0 = \alpha_1 + \alpha_2. \tag{6}$$

The second initial condition follows from the equality $u = v_R + v_C + v_L$ (see Example 4.4.2(b)), or

$$v_L(0) = u(0) - v_R(0) - v_C(0).$$

Since $u(0) = 1$, $v_R(0) = Ri(0) = Ry(0) = 0$, and because the charge of the capacitor at time zero $q(0) = Cv_C(0)$ is zero we have also $v_C(0) = 0$, it follows that

$$v_L(0) = 1.$$

Since $v_L = Ldi/dt = Ldy/dt$ it follows that the second initial condition we are looking for is

$$y^{(1)}(0) = \frac{1}{L}v_L(0) = \frac{1}{L}.$$

Differentiation of (5) with respect to time and setting $t = 0$ accordingly leads to

$$\frac{1}{L} = \alpha_1\lambda_1 + \alpha_2\lambda_2. \tag{7}$$

The two equations (6) and (7) may be solved for the constants α_1 and α_2, resulting in

$$\alpha_1 = -\alpha_2 = \frac{\frac{1}{L}}{\lambda_1 - \lambda_2}.$$

As a result, the solution of the differential equation for the given input and initial conditions is

$$y(t) = \frac{\frac{1}{L}}{\lambda_1 - \lambda_2}(e^{\lambda_1 t} - e^{\lambda_2 t}), \qquad t \geq 0.$$

To make this result more concrete, we adopt the numerical values $R = 11\ \Omega$, $L = 0.01$ H, and $C = 0.001$ F. As a result, the polynomial Q is given by $Q = \lambda^2 + 1100\lambda + 100000 = (\lambda + 100)(\lambda + 1000)$, so that the characteristic roots are $\lambda_1 = -100$ and $\lambda_2 = -1000$. After substitution of the numerical values we find that the solution of the differential equation is

$$y(t) = \frac{1}{9}(e^{-100t} - e^{-1000t}), \qquad t \geq 0.$$

A plot is given in Fig. 4.5.

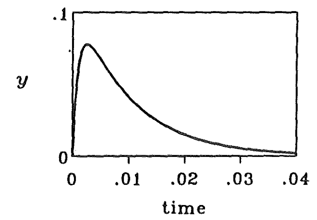


Figure 4.5. Response of the current through the RCL network when the input is a constant voltage equal to 1 and the initial charge of the capacitor and the initial flux through the inductor are zero. ∎

We review the results of this section for sampled difference systems.

4.4.8. Review: Homogeneous solutions of sampled linear time-invariant difference systems. Sampled linear time-invariant difference systems are described by difference equations of the form

$$Q(\sigma^T)y = P(\sigma^T)u,$$

where Q and P are polynomials as in (1). We study such systems on the semi-infinite time axis $\mathbb{T}_+ = \{t_o, t_o + T, t_o + 2T, \cdots\}$, with $t_o \in \mathbb{Z}(T)$, or on the infinite time axis $\mathbb{T}_\infty = \mathbb{Z}(T)$.

On the time axis $\mathbb{T}_+$ the homogeneous equation

$$Q(\sigma^T)y = 0$$

has N basis solutions $y_1, y_2, \cdots, y_N$, with N the degree of Q. On the time axis $\mathbb{T}_\infty$, the homogeneous equation has N_o basis solutions, with N_o the number of nonzero roots of Q. Every solution of the homogeneous equation may be expressed as a linear combination of the basis solutions.

Corresponding to each *nonzero* characteristic root λ of multiplicity m of the polynomial Q the homogeneous equation has m basis solutions given by

$$(t/T)^i \lambda^{t/T}, \qquad t \in \mathbb{T}_+ \text{ or } \mathbb{T}_\infty,$$

for $i = 0, 1, \cdots, m-1$. On the time axis $\mathbb{T}_+$ the homogeneous equation has the m_o *additional* basis solutions

$$\frac{1}{T}\Delta(t - t_o - iT), \qquad t \in \mathbb{T}_+,$$

for $i = 0, 1, \cdots, m_o - 1$, where $m_o = N - N_o$ is the number of *zero* roots of Q.

Any solution of the nonhomogeneous difference equation may be written as

$$y = y_{\text{part}} + y_{\text{hom}},$$

where y_{part} is a particular solution, and y_{hom} a suitable solution of the homogeneous equation. ∎

4.5 INITIALLY-AT-REST LINEAR TIME-INVARIANT DIFFERENTIAL AND DIFFERENCE SYSTEMS

In Section 4.3 we defined initially-at-rest difference and differential systems as difference and differential systems that are "at rest" at time $-\infty$. The system is at rest if a zero input from the initial time on results in a unique output that is identical to zero.

Initially-at-rest linear time-invariant difference and differential systems are input-output mapping systems. According to 4.3.14 they are *convolution* systems, characterized by an IO map of the form

$$y = h * u,$$

with h the *impulse response* of the system.

In this section we first consider how to determine the impulse response h of initially-at-rest linear difference and differential systems. Next it is seen how the impulse response may be used to generate particular solutions of difference and differential systems defined on infinite and semi-infinite time axes. Given a particular solution, the general solution of the difference or differential equation, or the response of the system to given initial conditions, may be found as in Section 4.4.

Impulse Response

We first discuss the impulse response of initially-at-rest difference systems described by a difference equation of the form

$$Q(\sigma)y = P(\sigma)u,$$

with Q and P the polynomials

$$Q(\lambda) = q_0 + q_1\lambda + \cdots + q_N\lambda^N,$$
$$P(\lambda) = p_0 + p_1\lambda + \cdots + p_M\lambda^M.$$

As before, the leading coefficients q_N and p_M are assumed to be nonzero.

The impulse response h of the initially-at-rest system is the response of the system if the input u is the unit pulse Δ. Because this input is zero for positive times for those times the impulse response h is a solution of the *homogeneous* equation and hence may be written as a linear combination of basis solutions on the time axis $\{1, 2, \cdots\}$. Because the input is zero for negative times as well and the system is initially at rest, the impulse response is zero for negative times up to some finite time, which is negative if the system is anticipating, and nonnegative if it is not anticipating. It may be shown that the impulse response of the difference system may actually be written in the form

$$h(n) = \sum_{i=1}^{N} \alpha_i y_i(n-1)\mathbb{1}(n-1) + \sum_{i=0}^{M-N} \beta_i \Delta(n+i), \qquad n \in \mathbb{Z}.$$

The α_i and β_i are suitable constants, and the y_i are basis solutions of the homogeneous difference equation on the semi-infinite time axis $\mathbb{Z}_+$, arbitrarily extended for negative times. If the lower limit of the second sum exceeds the upper limit (i.e., if $M - N < 0$), the sum is canceled.

In the continuous-time case, the system is described by the differential equation

$$Q(D)y = P(D)u,$$

and the impulse response h of the initially-at-rest system is its response to the input $u = \delta$. Again, because this input is zero for positive times, for those times the impulse response is a solution of the homogeneous equation, which means that for positive times it is a linear combination of the basis solutions. Also, because the input is zero for negative times and the system is initially at rest, the impulse response is zero for negative times. At time zero the impulse response has some delta functions, depending on whether $M - N \geq 0$. The general form of the impulse response for the continuous-time case is

$$h(t) = \sum_{i=1}^{N} \alpha_i y_i(t)\mathbb{1}(t) + \sum_{i=0}^{M-N} \beta_i \delta^{(i)}(t), \qquad t \in \mathbb{R},$$

where again the α_i and β_i are suitable constants, the y_i are basis solutions of the homogeneous differential equation, and the second sum is canceled if $M - N < 0$.

Solution for the unknown constants gives $\alpha_1 = -2/\sqrt{LC}$ and $\alpha_2 = 1/LC$, which yields the desired homogeneous solution. The impulse response follows as

$$h(t) = \left[-\frac{2}{\sqrt{LC}}e^{-t/\sqrt{LC}} + \frac{t}{LC}e^{-t/\sqrt{LC}}\right]\mathbb{1}(t) + \delta(t), \qquad t \in \mathbb{R}.$$

A plot of the impulse response is given in Fig. 4.7.

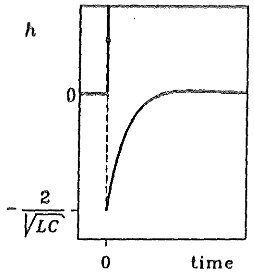


Figure 4.7. Impulse response of the RCL network for critical damping. ∎

Particular Solutions of Linear Time-Invariant Difference and Differential Systems

In Section 4.4 we introduced the notion of a *particular solution* of the difference equation $Q(\sigma)y = P(\sigma)u$ or the differential equation $Q(D)y = P(D)u$ corresponding to an input u. Given the impulse response h of the initially-at-rest system defined by the difference or differential equation, the following result makes it possible to construct particular solutions for a large class of inputs. Once a particular solution is available it is a simple matter to determine the general solution and the response to given initial conditions.

4.5.3. Summary: Particular solutions of linear constant coefficient difference and differential equations.

Let h be the impulse response of the initially-at-rest system described by the difference equation $Q(\sigma)y = P(\sigma)\,u$.

(a) On the infinite time axis $\mathbb{Z}$ the difference equation has the particular solution

$$y_{\text{part}} = h * u,$$

provided the convolution exists.

(b) On the semi-infinite time axis $\mathbb{T}_+ = \{n_o, n_o + 1, \cdots\}$ the difference equation has a particular solution which on $\mathbb{T}_+$ coincides with the signal

$$y_{\text{part}} = h * u_+,$$

provided the convolution exists, where u_+ is defined on the time axis $\mathbb{Z}$ by

$$u_+(n) = \begin{cases} u(n) & \text{for } n \geq n_o, \\ 0 & \text{otherwise,}\end{cases} \qquad n \in \mathbb{Z}.$$

Let h be the impulse response of the initially-at-rest system described by the differential equation $Q(D)y = P(D)u$.

(a′) On the infinite time axis $\mathbb{R}$ the differential equation has the particular solution

$$y_{\text{part}} = h * u,$$

provided the convolution exists.

(b′) On the semi-infinite time axis $\mathbb{T}_+ = [t_o, \infty)$ the differential equation has a particular solution which on $\mathbb{T}_+$ coincides with the signal

$$y_{\text{part}} = h * u_+,$$

provided the convolution exists, where u_+ is defined on the time axis $\mathbb{R}$ by

$$u_+(t) = \begin{cases} u(t) & \text{for } t \geq t_o, \\ 0 & \text{otherwise,}\end{cases} \qquad t \in \mathbb{R}.$$ ∎

4.5.4. Remark: Particular solutions. If the convolutions in 4.5.3 do not exist, both in the discrete- and the continuous-time case one may try to find particular solutions of the form

$$y_{\text{part}} = \left(h + \sum_i \alpha_i y_i\right) * u$$

on the infinite time axis $\mathbb{T}$, or

$$y_{\text{part}} = \left(h + \sum_i \alpha_i y_i\right) * u_+$$

on the semi-infinite time axis $\mathbb{T}_+$. The y_i are basis solutions on the infinite time axis $\mathbb{T}_\infty$ and the α_i constants that are chosen such that the convolutions exist. ∎

4.5.5. Proof of 4.5.3 and 4.5.4. The proof is given for the discrete-time case and a semi-infinite time axis. The proof for the continuous-time case follows parallel lines and the proof for infinite time axes is obtained by replacing n_o or t_o with $-\infty$.

Using the linearity and shift properties of the convolution it follows that

$$Q(\sigma)y_{\text{part}} = Q(\sigma)\left(\left(h + \sum_i \alpha_i y_i\right) * u_+\right) = \left(Q(\sigma)h + \sum_i \alpha_i Q(\sigma)y_i\right) * u_+.$$

Because y_i is a basis solution of the homogeneous equation, $Q(\sigma)y_i = 0$. Furthermore, since h is the impulse response, $Q(\sigma)h = P(\sigma)\Delta$. As a result,

$$Q(\sigma)y_{\text{part}} = (P(\sigma)\Delta) * u_+ = P(\sigma)(\Delta * u_+) = P(\sigma)u_+.$$

By restricting this equality to the time axis $\mathbb{T}_+$ we may replace u_+ with u, which proves that y_{part} satisfies the difference equation on $\mathbb{T}_+$. ∎

It follows from 4.5.3 that both in the discrete- and the continuous-time case the response of a difference or differential system on the infinite time axis $\mathbb{T}$ may be expressed as

$$y = h * u + y_{\text{hom}},$$

where y_{hom} is a suitable solution of the homogeneous equation. The first term $h * u$ is the *initially-at-rest* response of the system to the input u. The second term y_{hom} is the response if the input were zero and, hence, is the *zero-input* response. Thus we may write

$$y = y_{\text{initially-at-rest}} + y_{\text{zero-input}}.$$

The following example illustrates how 4.5.3 may be used to construct particular and general solutions.

4.5.6. Example: Particular solutions for the exponential smoother. As found in 4.4.2(a), for $a \neq 0$ the exponential smoother has the single basis solution

$$y_1(n) = a^n, \qquad n \in \mathbb{T}_+ \text{ or } \mathbb{T}_\infty,$$

while from Example 4.3.12 its impulse response is

$$h(n) = (1 - a)a^n \mathbb{1}(n), \qquad n \in \mathbb{Z}.$$

We consider the problem of finding a particular solution corresponding to the constant input

$$u(n) = 1, \qquad n \in \mathbb{T},$$

for different choices of $\mathbb{T}$ and a.

(a) $\mathbb{T} = \mathbb{T}_+ = \{n_o, n_o + 1, n_o + 2, \cdots\}$. We first consider the smoother on the semi-infinite time axis $\mathbb{T}_+ = \{n_0, n_0 + 1, n_0 + 2, \cdots\}$. Defining u_+ as

$$u_+(n) = \begin{cases} 1 & \text{for } n \geq n_o, \\ 0 & \text{otherwise}, \end{cases} \qquad n \in \mathbb{Z},$$

it follows from 4.5.3 that a particular solution corresponding to the input u coincides on $\mathbb{T}_+$ with

$$\begin{aligned} y_{\text{part}}(n) &= (h * u_+)(n) = \sum_{m=-\infty}^{\infty} h(n - m)u_+(m) \\ &= (1 - a) \sum_{m=n_o}^{\infty} a^{n-m}\mathbb{1}(n - m), \qquad n \in \mathbb{Z}. \end{aligned}$$

For $n \geq n_o$ we have

$$y_{\text{part}}(n) = (1 - a) \sum_{m=n_o}^{n} a^{n-m} = 1 - a^{n-n_o+1}, \qquad n \geq n_o, n \in \mathbb{Z}. \tag{3}$$

This is the desired particular solution. It follows that the *general* solution of the difference equation on the time axis $\mathbb{T}_+$ is given by

$$y(n) = 1 - a^{n-n_o+1} + \alpha a^{n-n_o}, \qquad n = n_o, n_o + 1, n_o + 2, \cdots, \tag{4}$$

with α an arbitrary constant.

(b) $\mathbb{T} = \mathbb{Z}$ *and* $|a| < 1$. On the infinite time axis $\mathbb{T}_\infty = \mathbb{Z}$ a particular solution follows by taking $n_o = -\infty$ in (3). This is well-defined if $|a| < 1$ and results in

$$y_{\text{part}}(\text{n}) = 1, \qquad \text{n} \in \mathbb{Z}. \tag{5}$$

(c) $\mathbb{T} = \mathbb{Z}$ *and* $|a| \geq 1$. When the time axis is $\mathbb{Z}$, but $|a| \geq 1$, (3) has no limit for $n_o \to -\infty$. Obviously, though, the constant solution (5) is a correct particular solution of the difference equation $y(n + 1) - ay(n) = (1 - a)$, $n \in \mathbb{Z}$, for *any* a. To find this solution constructively when $|a| \geq 1$, we use 4.5.4. Consider $h' = h + \alpha_1 y_1$, with α_1 to be chosen. We have

$$h'(n) = (1 - a)a^n\mathbb{1}(n) + \alpha_1 a^n, \qquad n \in \mathbb{Z}.$$

By choosing $\alpha_1 = a - 1$ this reduces to

$$\begin{aligned} h'(n) &= \begin{cases} (a - 1)a^n & \text{for } n < 0, \\ 0 & \text{for } n \geq 0 \end{cases} \\ &= (a - 1)a^n\mathbb{1}(-n - 1), \qquad n \in \mathbb{Z}. \end{aligned}$$

For $|a| \geq 1$ the function h' has finite action, so that its convolution with the constant input u exists. It is easily found that

$$(h' * u)(n) = 1, \qquad n \in \mathbb{Z},$$

which confirms that also for $|a| \geq 1$ the constant 1 is a particular solution of the difference equation. ∎

We very briefly review the results of this section for sampled systems.

4.5.7. Review: Impulse response and particular solutions of sampled difference systems. The impulse response of initially-at-rest sampled difference systems defined on the time axis $\mathbb{Z}(T)$ and described by the difference equation $Q(\sigma^T)y = P(\sigma^T)u$ is of the form

$$h(t) = \sum_{l=1}^{N} \alpha_l y_l(t - T)\mathbb{1}(t - T) + \sum_{l=0}^{M-N} \beta_l \Delta\left(\frac{t + iT}{T}\right), \qquad t \in \mathbb{Z}(T),$$

where the y_l are basis solutions on the time axis $\mathbb{Z}(T)$, and the α_l and β_l suitable constants. Particular solutions of the difference equation may be generated by convolution with the impulse response as in 4.5.3(a) and (b) and 4.5.4. ∎

4.6 STABILITY OF DIFFERENCE AND DIFFERENTIAL SYSTEMS

In 3.6.1 we defined *BIBO* (bounded-input bounded-output) stability of *convolution* systems. A convolution system is BIBO stable if every bounded input results in a bounded output. In the present section we extend the notion of BIBO stability first to IOM systems, and then to IO systems, which do not have a unique output for each input. We also introduce a stronger form of stability, namely, *CICO* (converging-input converging-output) stability. An IO or IOM system is CICO stable if first of all it is BIBO stable, and furthermore its responses to two inputs that converge to each other also converge to each other.

For linear time-invariant difference and differential systems simple conditions for the various types of stability may be obtained in terms of their characteristic roots and *poles*. First, the latter notion is defined.

4.6.1. Definition: Poles and zeros. Consider the constant coefficient difference system

$$Q(\sigma)y = P(\sigma)u$$

or the constant coefficient differential system

$$Q(D)y = P(D)u.$$

The roots of the polynomial Q are the *characteristic roots* of the system. Let P' and Q' be the polynomials that are obtained by canceling all common polynomial factors of P and Q. Then the roots of Q' are called the *poles* of the system, and those of P' the *zeros* of the system. ∎

We note that the rational function H defined by

$$H = \frac{P}{Q}$$

is *infinite* at the poles of the system, while at the zeros of the system it is *zero*. In Chapter 8 the function H is re-encountered as the *transfer function* of the system.

4.6.2. Example: Poles and zeros. Consider the difference or differential system characterized by the polynomials

$$Q(\lambda) = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2),$$
$$P(\lambda) = \lambda^2 + 2\lambda - 3 = (\lambda - 1)(\lambda + 3).$$

From Q we see that the system has the characteristic roots 1 and -2. After cancellation of the common factor $\lambda - 1$ we obtain the polynomials

$$Q'(\lambda) = \lambda + 2, \qquad P'(\lambda) = \lambda + 3.$$

As a result, the system has a single pole -2 and a single zero -3. ∎

BIBO Stability of Initially-At-Rest Linear Time-Invariant Difference and Differential Systems

Initially-at-rest difference and differential systems are convolution systems defined on the infinite time axis $\mathbb{Z}$ or $\mathbb{R}$, respectively. BIBO stability of convolution systems was defined in 3.6.1. In 3.6.2 we saw that a convolution system is BIBO stable if and only if its impulse response h has finite action $\|h\|_1$. For initially-at-rest difference and differential systems we may formulate necessary and sufficient conditions for BIBO stability in terms of their poles.

4.6.3. Summary: BIBO stability of initially-at-rest linear time-invariant difference and differential systems.

The initially-at-rest difference system characterized by the constant coefficient difference equation

$$Q(\sigma)y = P(\sigma)u,$$

is BIBO stable if and only if all its poles have magnitude strictly less than 1.

The initially-at-rest differential system characterized by the constant coefficient differential equation

$$Q(D)y = P(D)u,$$

is BIBO stable if and only if

(i) the degree of P is less than or equal to that of Q, and

(ii) all the poles of the system have strictly negative real part. ∎

The proof follows by inspection of the impulse response h of the initially-at-rest system, which by 4.5.1 depends on the basis solutions of the homogeneous equation. If some of the poles have magnitude greater than or equal to 1 (in the discrete-time case) or real part greater than or equal to zero (in the continuous-time case) the corresponding basis solutions do not converge to zero. As a result the impulse response has infinite action $\|h\|_1$, and by 3.4.15 the system is BIBO unstable. Characteristic roots that are *not* poles, that is, characteristic roots that cancel against roots of P, do *not* appear in the impulse response, and therefore do not affect the BIBO stability of the system. The details are given in E.1 in Supplement E. In this supplement a number of proofs have been collected. If in the continuous-time case the degree of P exceeds that of Q, the impulse response contains derivative δ-functions, which make the action infinite and hence render the convolution system BIBO unstable.

4.6.4. Example: BIBO stability of an initially-at-rest differential system. From Example 4.6.2 it follows that the differential system

$$y^{(2)}(t) + y^{(1)}(t) - 2y(t) = u^{(2)}(t) + 2u^{(1)}(t) - 3u(t), \qquad t \in \mathbb{R},$$

has the two characteristic roots 1 and -2 but a single pole -2. By 4.6.3 the corresponding initially-at-rest system is BIBO stable even though one characteristic root is positive. Indeed, it may be found that the impulse response of the system is

$$h(t) = \delta(t) + e^{-2t}\mathbb{1}(t), \qquad t \in \mathbb{R}, \tag{1}$$

which does not contain a term e^t corresponding to the characteristic root 1, and has finite action. *Exercise:* Prove that the system has the impulse response (1). ∎

Boundedness and Convergence of Zero Input Responses

Before plunging into a detailed exposition of the stability of input-output systems we consider the boundedness and convergence of their *zero-input response,* that is, the solution of the homogeneous difference or differential equation $Q(\sigma)y = 0$ or $Q(D)y = 0$, respectively. The results apply both to semi-infinite and infinite time axes.

4.6.5. Summary: Boundedness and convergence of homogeneous solutions.

(a) Necessary and sufficient conditions for any solution of the homogeneous difference equation

$$Q(\sigma)y = 0$$

to remain bounded from any finite time on are that

(i) all characteristic roots have magnitude less than or equal to 1 and

(ii) any characteristic root with magnitude equal to 1 has multiplicity one.

(b) Necessary and sufficient conditions for any solution of the homogeneous difference equation

$$Q(\sigma)y = 0$$

to converge to zero as time increases to infinity are that all characteristic roots have magnitude strictly less than 1.

(a') Necessary and sufficient conditions for any solution of the homogeneous differential equation

$$Q(D)y = 0$$

to remain bounded from any finite time on are that

(i) all characteristic roots have nonpositive real part and

(ii) any characteristic root with zero real part has multiplicity one.

(b') Necessary and sufficient conditions for any solution of the homogeneous differential equation

$$Q(D)y = 0$$

to converge to zero as time increases to infinity are that all characteristic roots have strictly negative real part.

The proof follows by the fact that any solution of the homogeneous equation is a linear combination of basis solutions. By 4.4.1 we may select basis solutions that in the discrete-time case are of the form

$$n^k\lambda^n = n^k\rho^n[\cos(\psi n) + j\sin(\psi n)], \qquad n \in \mathbb{Z},$$

where λ is a characteristic root, $\rho = |\lambda|$ and $\psi = \arg(\lambda)$. In the continuous-time case we choose the basis solutions

$$t^k e^{\lambda t} = t^k e^{\sigma t}[\cos(\omega t) + j\sin(\omega t)], \qquad t \in \mathbb{R},$$

where again λ is a characteristic root, $\sigma = \mathrm{Re}(\lambda)$ and $\omega = \mathrm{Im}(\lambda)$. The basis solutions, and hence any linear combination of the basis solutions, are bounded or converge to zero as time increases under the conditions stated in 4.6.5.

4.6.6. Example: Convergence of the zero-input response of a differential system. The system of Example 4.6.4, which is described by the differential equation

$$y^{(2)}(t) + y^{(1)}(t) - 2y(t) = u^{(2)}(t) + 2u^{(1)}(t) - 3u(t), \qquad t \in \mathbb{R},$$

has the characteristic roots 1 and -2. Because the first of these has positive real part, by 4.6.5 the zero-input response generally does not converge to zero. Indeed, the zero-input response is any linear combination of the exponentials e^t and e^{-2t}, $t \in \mathbb{R}$, the first of which goes to infinity as time increases. ∎

We are now in a position to discuss the stability of *IO* systems, beginning with their BIBO stability and proceeding to CICO stability.

BIBO Stability of IO Systems

BIBO stability of convolution systems is the property that bounded inputs result in bounded outputs. For general input-output systems the definition need be slightly modified, as an example will illustrate. We first give the definition.

4.6.7. Definition: BIBO stability of IO systems. An input-output system with scalar input and output defined on the infinite or right semi-infinite time axis $\mathbb{T}$ is *BIBO stable* if for any input-output pair (u, y)

$$\|u\|_\infty < \infty \quad \text{implies} \quad \|y_\theta\|_\infty < \infty$$

for every $\theta \in \mathbb{T}$, where

$$y_\theta(t) = \begin{cases} y(t) & \text{for } t \geq \theta, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{T}. \qquad ■$$

Figure 4.8 illustrates the signal y_θ. The definition implies that an IO system is BIBO stable if any bounded input always produces an output that is bounded *from any finite time on*.

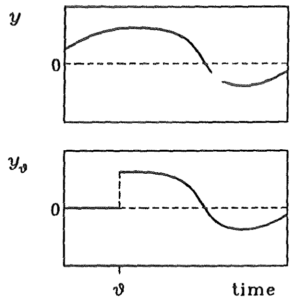


Figure 4.8. The signals y and y_θ.

First we show why the definition of BIBO stability must be modified for IO systems.

4.6.8. Example: BIBO stability of the exponential smoother as an IO system. The exponential smoother is described by the difference equation

$$y(n+1) = ay(n) + (1-a)u(n+1), \qquad n \in \mathbb{Z}.$$

The impulse response h of the smoother is given by

$$h(n) = (1-a)a^n \mathbb{1}(n), \qquad n \in \mathbb{Z}.$$

Suppose that $|a| < 1$; then h has finite action and the convolution $h * u$ exists for any input u with finite amplitude. By 4.5.3(a) any output y corresponding to a bounded input u may be written as

$$y(n) = (h * u)(n) + \alpha a^n, \qquad n \in \mathbb{Z},$$

with α an arbitrary constant. Because h has finite action and u is bounded, the first term on the right-hand side is bounded. The second term is always bounded from any finite time on, even though it increases to infinity in the negative time direction and hence is not bounded. By Definition 4.6.7 this system is still called BIBO stable. ■

Because IOM systems are IO systems, Definition 4.6.7 also applies to IOM systems and, in particular, also to convolution systems. It is easily seen that BIBO stability of convolution systems as defined in 3.6.1 implies BIBO stability in the sense of 4.6.7. By a slight modification of the proof of 3.6.2, it may be shown that convolution systems are BIBO stable in the sense of 4.6.7 if and only if their impulse response h has finite action $\|h\|_1$. Hence, BIBO stability of convolution systems in the sense of 3.6.1 is equivalent to BIBO stability in the sense of 4.6.7.

4.6.9. Summary: BIBO stability of convolution systems. A discrete- or continuous-time convolution system with impulse response h is BIBO stable (in the sense of 4.6.7) if and only if h has finite action, that is, $\|h\|_1 < \infty$. ■

We next consider the BIBO stability of IO systems described by constant coefficient difference and difference equations. Obviously, a necessary condition for the BIBO stability of such systems is that their impulse response has bounded action; otherwise there exist bounded inputs that result in unbounded initially-at-rest responses. From 4.6.3 we know that a necessary and sufficient condition for the impulse response to have bounded action in the discrete-time case is that all the poles of the system have magnitude strictly less than one. In the continuous-time case the degree of Q should not be less than that of P, and the poles should have strictly negative real part.

Suppose that these conditions are satisfied, so that the impulse response h of the system has finite action. Then by 4.5.3 the response to a bounded input u may be written as

$$y = h * u + \sum_i \alpha_i y_i,$$

where the y_i are basis solutions and the α_i arbitrary constants. Because by assumption h has finite action and u is bounded, the first term is bounded. Necessary and sufficient conditions for the second term to be bounded are given in 4.6.5. Since by assumption the poles of the system have magnitude strictly less than one (in the discrete-time case) or strictly negative real part (in the continuous-time case), for the boundedness of the second term we need only look at those characteristic roots that are *not* poles (i.e., that cancel against roots of P). Combining the results we conclude what follows.

4.6.10. Summary: BIBO stability of constant coefficient linear difference and differential systems.

Necessary and sufficient conditions for the IO system described by the constant coefficient linear difference equation

$$Q(\sigma)y = P(\sigma)u$$

to be BIBO stable are the following:

Necessary and sufficient conditions for the IO system described by the constant coefficient linear differential equation

$$Q(D)y = P(D)u$$

to be BIBO stable are the following:

(i) all the poles of the system have magnitude strictly less than 1,

(ii) the system has no canceled characteristic roots with magnitude strictly greater than 1, and

(iii) any canceled characteristic root with magnitude equal to 1 has multiplicity one.

(i) the degree of P is less than or equal to that of Q,

(ii) all the poles of the system have strictly negative real part,

(iii) the system has no canceled characteristic roots with strictly positive real part, and

(iv) any canceled characteristic root with zero real part has multiplicity one. ∎

This result applies to systems defined on both semi-infinite and infinite time axes.

4.6.11. Example: BIBO unstable differential system. The initially-at-rest differential system described by the differential equation

$$y^{(2)}(t) + y^{(1)}(t) - 2y(t) = u^{(2)}(t) + 2u^{(1)}(t) - 3u(t), \qquad t \in \mathbb{R},$$

was in Example 4.6.4 found to be BIBO stable. Because the impulse response h of the system has finite action, by 4.5.3 for bounded inputs the general solution of the differential equation is given by

$$y(t) = (h * u)(t) + \alpha_1 e^t + \alpha_2 e^{-2t}, \qquad t \in \mathbb{R},$$

with h the impulse response (1), and α_1 and α_2 arbitrary constants. The first term on the right-hand side is bounded if the input u is bounded, but the second term generally is unbounded in the positive time direction. Hence, the system is not BIBO stable. Indeed, the system has the canceled characteristic root 1, which violates the conditions of 4.6.10. ∎

CICO Stability

A form of stability that is stronger than BIBO stability is *converging-input converging-output* (CICO) stability. The idea of CICO stability may be explained as follows. Suppose that an IO or IOM system is subjected to a bounded input u that approaches zero as time increases, that is, $u(t) \to 0$ as $t \to \infty$. Then, if the system is BIBO stable, all we know is that the corresponding output is bounded. It is easy to think of systems, however, that have the additional property that if the input approaches zero, any corresponding output y also approaches zero, (i.e., if the input eventually comes to rest, so does the output). Such systems are called CICO stable.

The formal definition of CICO stability is slightly more involved.

4.6.12. Definition: CICO stability. Let (u_1, y_1) and (u_2, y_2) be two input-output pairs of an IO system, which is defined on the infinite or right semi-infinite time axis $\mathbb{T}$, such that $\|u_1 - u_2\|_\infty < \infty$. Then the system is *converging-input converging-output* (CICO) *stable* if

(i) $\|(y_1 - y_2)_\theta\|_\infty < \infty$ for every $\theta \in \mathbb{T}$,

(ii) $|u_1(t) - u_2(t)| \to 0$ as $t \to \infty$ implies $|y_1(t) - y_2(t)| \to 0$ as $t \to \infty$. ∎

The notation $(y_1 - y_2)_\theta$ is as in 4.6.7. Fig. 4.9 illustrates CICO stability. Note that (i) implies that any CICO system is also BIBO stable (by setting $u_2 = 0$.) The converse, namely, that BIBO stable IO systems are also CICO stable, is not true in general. It does hold for BIBO stable *convolution* systems, however.

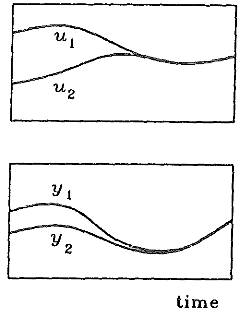


Figure 4.9. CICO stability. Top: two inputs approach each other. Bottom: the two corresponding outputs also approach each other.

4.6.13. Summary: CICO stability of convolution systems. A discrete- or continuous-time convolution system is CICO stable if and only if it is BIBO stable. ∎

The proof is presented in E.2 in Supplement E.

We consider the CICO stability of constant coefficient difference and differential systems. Let (u_1, y_1) and (u_2, y_2) be two input-output pairs such that $u_1 - u_2$ is bounded. If the impulse response h of the system has finite action, by linearity,

$$y_1 - y_2 = h * (u_1 - u_2) + \sum_i \alpha_i y_i,$$

where the y_i are basis solutions, and the α_i constants. By 4.6.13, the first term on the right-hand side converges to zero as time increases if $u_1 - u_2$ converges to zero. By 4.6.5, the second term converges to zero as time increases if and only if all characteristic roots of the system have magnitude strictly less than 1 (in the discrete-time case) or strictly negative real part (in the continuous-time case.) These conditions are also sufficient for the impulse response to have finite action (plus, in the continuous-time case, the requirement that the impulse response contain no derivatives of delta functions) and, hence, imply CICO stability.

The conclusions apply to systems defined both on semi-infinite and infinite time axes, and may be summarized as follows.

4.6.14. Summary: CICO stability of constant coefficient difference and differential systems.

Necessary and sufficient conditions for the IO system described by the constant coefficient difference equation	Necessary and sufficient conditions for the IO system described by the constant coefficient differential equation
$Q(\sigma)y = P(\sigma)u$	$Q(D)y = P(D)u$
to be CICO stable are that all the characteristic roots of the system have magnitude strictly less than 1.	to be CICO stable are that (i) the degree of Q is greater than or equal to that of P and (ii) all the characteristic roots of the system have strictly negative real part. ∎

Note that the conditions for CICO stability of constant coefficient difference and differential systems are the same as the conditions of 4.6.5 for the convergence of the zero-input response.

CICO stability is the strongest form of stability of constant coefficient linear difference and differential systems. It implies, but is not implied by, BIBO stability.

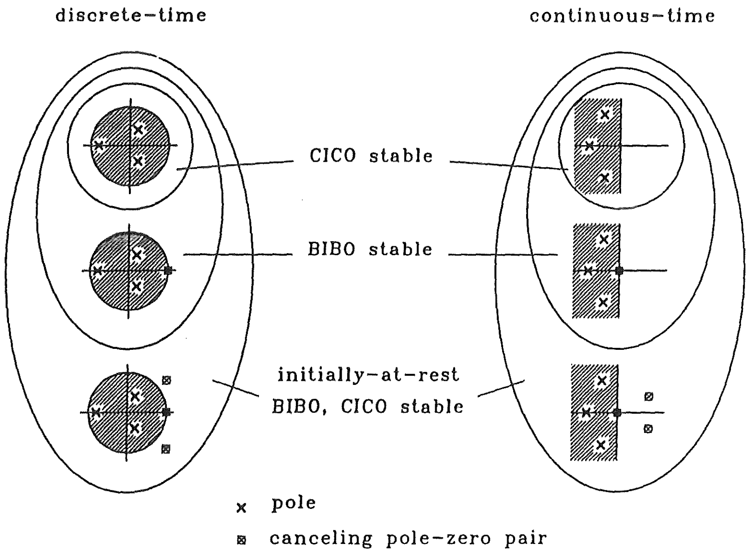


Figure 4.10. Interrelation of various forms of stability of constant coefficient linear difference and differential systems, and examples of pole-zero patterns.

BIBO stability, in turn, implies but is not implied by BIBO and CICO stability of the *initially-at-rest* system. Figure 4.10 illustrates that the set of all initially-at-rest BIBO and CICO stable constant coefficient linear difference and differential systems contains the set of all BIBO stable systems, which in turn contains the set of all CICO stable systems. The figure also shows examples of pole-zero patterns for each form of stability.

4.6.15. Example: CICO stability.

(a) *Exponential smoother*. From the difference equation

$$y(n+1) - ay(n) = (1-a)u(n+1), \qquad n \in \mathbb{Z},$$

that describes the exponential smoother we see that the system has a single characteristic root a. Hence, the smoother is CICO stable if and only if $|a| < 1$.

(b) *Differential system*. In Example 4.6.4 we saw that the differential system

$$y^{(2)}(t) + y^{(1)}(t) - 2y(t) = u^{(2)}(t) + 2u^{(1)}(t) - 3u(t), \qquad t \in \mathbb{R},$$

has the two characteristic roots 1 and -2. Hence, by 4.6.14 the system is not CICO stable. We note that according to 4.6.4 the *initially-at-rest* differential system is BIBO and hence also CICO stable. The *IO* system, however, is neither BIBO nor CICO stable. The reason is that because of the characteristic root 1 the response of the IO system generally has an unstable component of the form e^t, $t \in \mathbb{R}$. ∎

We conclude, as usual, with a review of the results of this section applied to sampled difference systems.

4.6.16. Review: Stability of sampled constant coefficient difference systems. Sampled constant coefficient difference systems are discrete-time systems on the time axis $\mathbb{Z}(T)$ described by the difference equation

$$Q(\sigma^T)y = P(\sigma^T)u, \tag{2}$$

with Q and P polynomials. The *characteristic roots* of the system are the roots of Q. The *poles* of the system are those characteristic roots that do not cancel against roots of P.

The initially-at-rest system defined by the difference equation (2) is BIBO stable if and only if all the poles of the system have magnitude strictly less than 1.

BIBO and CICO stability of sampled discrete-time systems are defined as in 4.6.7 and 4.6.12, respectively. The conditions for BIBO and CICO stability of sampled constant coefficient difference IO systems are identical to those given in 4.6.10 and 4.6.14 for difference systems defined on the time axis $\mathbb{Z}$. ∎

4.7 FREQUENCY RESPONSE OF DIFFERENCE AND DIFFERENTIAL SYSTEMS

This section is devoted to an investigation of the response of linear constant coefficient difference and differential systems to *harmonic* inputs. As before, the initially-at-rest system plays a central role, and is analyzed first.

The initially-at-rest system is a convolution system. As shown in Section 3.7, the response of a convolution system to harmonic inputs is determined by its *frequency response function*. It is seen in the present section that the frequency response function of difference and differential systems may directly be obtained from the polynomials Q and P that define the difference or differential equation.

Once the frequency response function is known it is not difficult to find the general response of difference and differential IO systems to harmonic inputs.

Frequency Response Function of Difference and Differential Systems

In Section 3.7 we found that the response of a convolution system to a harmonic input, if it exists, is again harmonic. If the system has the impulse response h with finite action $\|h\|_1$, then its frequency response function $\hat{h}$ is given by

$$\hat{h}(f) = \sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn}, \qquad f \in \mathbb{R},$$

in the discrete-time case, and

$$\hat{h}(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt, \qquad f \in \mathbb{R},$$

in the continuous-time case. Given the frequency response function, we have the respective harmonic input-output pairs

$$\text{discrete-time:} \quad u(n) = e^{j2\pi fn}, \quad y(n) = \hat{h}(f)e^{j2\pi fn}, \qquad n \in \mathbb{Z},$$

$$\text{continuous-time:} \quad u(t) = e^{j2\pi ft}, \quad y(t) = \hat{h}(f)e^{j2\pi ft}, \qquad t \in \mathbb{R}.$$

The frequency response function of initially-at-rest difference and differential systems may immediately be determined from the difference or differential equation, without first computing the impulse response.

4.7.1. Summary: Frequency response function of initially-at-rest difference and differential systems.

The frequency response function $\hat{h}$ of the initially-at-rest difference system with difference equation discreto

$$Q(\sigma)y = P(\sigma)u$$

exists if and only if all the poles of the system have magnitude strictly less than one and is given by

$$\hat{h}(f) = \frac{P(e^{j2\pi f})}{Q(e^{j2\pi f})}, \qquad f \in \mathbb{R}.$$

The frequency response function $\hat{h}$ of t initially-at-rest differential system with di ferential equation continuo

$$Q(D)y = P(D)u$$

exists if and only if all the poles of the sys tem have strictly negative real part, and i given by

$$\hat{h}(f) = \frac{P(j2\pi f)}{Q(j2\pi f)}, \qquad f \in \mathbb{R}.$$

■

4.7.2 Proof. The proof is given for the continuous-time case and consists of two steps. First we establish the existence of the frequency response function, and then we derive the function itself.

(i) *Existence.* From 4.5.1 the impulse response h of the initially-at-rest system is the sum of a δ-function and some derivative δ-functions and a linear combination of basis solutions multiplied by the unit step. The δ-functions do not affect the existence of the frequency response function. The basis solutions that are involved are of the form $t^k e^{\lambda t}$, $t \in \mathbb{R}$, where λ is a pole of the system. Characteristic roots λ that are not poles do not appear in h. If the poles all have strictly negative real parts, each of the basis solutions decays exponentially. As a result, the nonsingular part of the impulse response has finite action so that by 3.7.2 the frequency response function of the system exists. On the other hand, if one or several of the poles have zero or positive real parts the impulse response does not converge to zero, the integral that defines the frequency response function diverges, and the frequency response function does not exist, at least not in the regular sense.

(ii) *Derivation of the frequency response function.* From Section 3.7 we know that the response of a continuous-time convolution system to the harmonic input $u(t) = e^{j2\pi ft}$, $t \in \mathbb{R}$, if it exists, is again harmonic and of the form $y(t) = y_o e^{j2\pi ft}$, with $y_o = \hat{h}(f)$. The existence of y is guaranteed if all the poles of the system have strictly negative real parts, as established in (i). All we have to do to find y_o is to substitute this solution into the differential equation. Since $n$fold differentiation of harmonic signals amounts to multiplication by $(j2\pi f)^n$, it follows after substitution of u and y into the differential equation $Q(D)y = P(D)u$ that

$$Q(j2\pi f)y_o e^{j2\pi ft} = P(j2\pi f)e^{j2\pi ft}, \qquad t \in \mathbb{R}.$$

Solution for y_o yields $y_o = \hat{h}(f) = P(j2\pi f)/Q(j2\pi f)$.

The proof for the discrete-time case follows the same lines, with the difference that the factor $j2\pi f$ in (ii) is replaced with $e^{j2\pi f}$. ∎

4.7.3. Example: Frequency response function of difference and differential systems.

(a) *Exponential smoother.* The exponential smoother is described by the difference equation

$$y(n+1) - ay(n) = (1-a)u(n+1), \qquad n \in \mathbb{Z},$$

so that the polynomials Q and P are given by

$$Q(\lambda) = \lambda - a, \qquad P(\lambda) = (1-a)\lambda.$$

If $a \neq 1$, the smoother has the single pole a. By 4.7.1, the frequency response function of the smoother exists if $|a| < 1$, and then is given by

$$\begin{aligned}\hat{h}(f) &= \frac{P(e^{j2\pi f})}{Q(e^{j2\pi f})} = \frac{(1-a)e^{j2\pi f}}{e^{j2\pi f} - a} \\ &= \frac{1-a}{1 - a\, e^{-j2\pi f}}, \qquad f \in \mathbb{R}.\end{aligned}$$

This is the same as the result we found in Example 3.7.5(a) with considerably more effort after first determining the impulse response h of the system. *Exercise:* Show that if $a = 1$ the frequency response function is $\hat{h} = 0$.

(b) *RCL network.* From Example 4.2.4(c) the differential equation that describes the RCL network with the voltage across the inductor as output is

$$\frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{d^2u(t)}{dt^2}, \qquad t \in \mathbb{R}.$$

The polynomials Q and P for this differential equation are given by

$$Q(\lambda) = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC},$$

$$P(\lambda) = \lambda^2.$$

The poles of the system are the characteristic roots

$$\lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}.$$

If $R^2 \geq 4L/C$, both poles are real and negative (assuming that R, C and L are all positive.) If $R^2 < 4L/C$, the poles form a conjugate pair with real part $-R/2L$. In either case, the poles have strictly negative real parts, so that the frequency response function $\hat{h}$ exists. It is given by

$$\hat{h}(f) = \frac{P(j2\pi f)}{Q(j2\pi f)} = \frac{-4\pi^2 f^2}{\left(\frac{1}{LC} - 4\pi^2 f^2\right) + \frac{R}{L}j2\pi f}, \qquad f \in \mathbb{R}.$$

In terms of the resonance frequency $\omega_r = 1/\sqrt{LC}$ and the quality factor $q = \omega_r L/R$ of the network this may be rewritten as

$$\hat{h}(\omega/2\pi) = \frac{-\omega^2/\omega_r^2}{(1 - \omega^2/\omega_r^2) + j\omega/\omega_r q}, \qquad \omega \in \mathbb{R},$$

where for convenience we use the angular frequency $\omega = 2\pi f$ rather than the frequency f. The magnitude and phase of $\hat{h}$ determine the response of the initially-at-rest RCL network to real harmonics, and are given by

$$|\hat{h}(\omega/2\pi)|^2 = \frac{\omega^4/\omega_r^4}{(1 - \omega^2/\omega_r^2)^2 + \omega^2/\omega_r^2 q^2},$$

$$\arg\left(\hat{h}(\omega/2\pi)\right) = \begin{cases} -\operatorname{atan}\left(\dfrac{\omega/\omega_r q}{1 - \omega^2/\omega_r^2}\right) + \pi & \text{for } 0 < \omega/\omega_r \leq 1, \\ -\operatorname{atan}\left(\dfrac{\omega/\omega_r q}{1 - \omega^2/\omega_r^2}\right) & \text{for } \omega/\omega_r > 1, \end{cases} \qquad \omega \in \mathbb{R}_+.$$

Plots of the magnitude and phase are given in Fig. 4.11 both for $q > 1/2$ and $q < 1/2$ on the normalized angular frequency scale ω/ω_r, $\omega \geq 0$. For $q > 1/2$ the characteristic roots of the network form a complex conjugate pair and as a result the impulse response has a damped oscillatory behavior. This is reflected by the presence of a peak in the magnitude of the frequency response function near the resonance frequency ω_r.

4.7.4. Remark: Poles on the unit circle or imaginary axis. If a constant coefficient linear difference system $Q(\sigma)y = P(\sigma)u$ has all its poles strictly inside the unit circle, the poles all have magnitude strictly less than one, so that by 4.7.1 the frequency response function of the system exists. If besides poles inside the unit circle the system also has poles *on* the unit circle, the frequency response function $\hat{h}$ of the system only exists in the *generalized* sense. It may be found by application of the generalized DCFT (see Section 7.4) to the impulse response of the system.

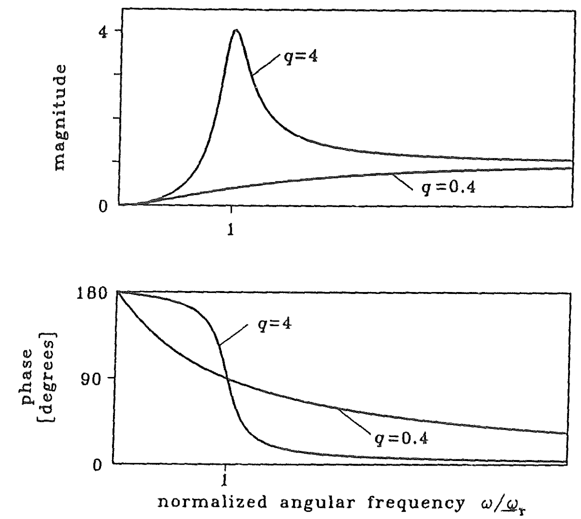


Figure 4.11. Frequency response function of the RCL network. Top: magnitude. Bottom: phase.

Likewise, a constant coefficient linear differential system has a frequency response function in the generalized sense if besides poles to the left of the imaginary axis it also has poles *on* the imaginary axis. The frequency response function may then be found by application of the generalized CCFT (see Section 7.5) to the impulse response. An *integrator*, for instance, is the differential system

$$y^{(1)}(t) = u(t), \qquad t \in \mathbb{R}.$$

Because $P(\lambda) = 1$ and $Q(\lambda) = \lambda$, the system has a single pole at 0 and, hence, by 4.7.1 the system does not have a frequency response function. Indeed, the corresponding initially-at-rest system has the IO map

$$y(t) = \int_{-\infty}^{t} u(\tau)\, d\tau, \qquad t \in \mathbb{R},$$

so that the impulse response of the system is

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \mathbb{1}(t), \qquad t \in \mathbb{R}.$$

The impulse response has infinite action, and hence the frequency response function does not exist in the ordinary sense.

As found in 7.5.5, the system has the *generalized* frequency response functi

$$\hat{h}(f) = \frac{1}{j2\pi f} + \tfrac{1}{2}\delta(f), \qquad f \in \mathbb{R}.$$

The first term on the right-hand side is the frequency response function that oı would expect from 4.7.1, but the frequency response function has an additional siı gularity at frequency zero (corresponding to the pole at 0).

Steady-State and Transient Response to Harmonic Inputs

We continue with a discussion of the response of difference and differential *IO* sys tems to harmonic inputs. It is easy to establish the following result.

4.7.5. Summary: Response of difference and differential systems to harmonic inputs. Any output of a CICO stable constant coefficient difference or differential system with frequency response function $\hat{h}$ corresponding to

(a) the complex harmonic input

$$u(t) = u_o e^{j2\pi f t}, \qquad t \in \mathbb{T},$$

or

(b) the real harmonic input

$$u(t) = u_o \cos(2\pi f t + \phi), \qquad t \in \mathbb{T},$$

with the time axis $\mathbb{T}$ infinite or right semi-infinite, is of the form

$$y = y_{\text{steady-state}} + y_{\text{transient}}.$$

The steady-state response $y_{\text{steady-state}}$ is given by

(a)

$$y_{\text{steady-state}}(t) = \hat{h}(f) u_o e^{j2\pi f t}, \qquad t \in \mathbb{T},$$

or

(b)

$$y_{\text{steady-state}}(t) = |\hat{h}(f)| u_o \cos(2\pi f t + \phi + \psi(f)), \qquad t \in \mathbb{T},$$

respectively, with $\psi(f) = \arg(\hat{h}(f))$. The transient response $y_{\text{transient}}$ is a solution of the homogeneous equation and satisfies

$$y_{\text{transient}}(t) \to 0 \qquad \text{as} \quad t \to \infty. \qquad ■$$

The proof for case (a) follows by observing that $\hat{h}(f)u_o e^{j2\pi ft}$, $t \in \mathbb{T}$, is a particular solution of the difference or differential equation for the given harmonic input. The general solution thus is given by

$$y(t) = \hat{h}(f)u_o e^{j2\pi ft} + y_{\text{hom}}(t), \qquad t \in \mathbb{T},$$

where y_{hom} is a solution of the homogeneous equation whose coefficients are determined by the initial conditions. Since by assumption the system is CICO stable, any solution of the homogeneous equation tends to zero. The proof for the real-valued harmonic input (b) is similar.

4.7.6. Example: Steady-state and transient response of the RC network. The RC network is described by the differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}u(t), \qquad t \in \mathbb{R}.$$

Because the system has the single pole $-1/RC$ the frequency response function of the system is well-defined and given by

$$\hat{h}(f) = \frac{1/RC}{j2\pi f + 1/RC} = \frac{1}{j2\pi fRC + 1}, \qquad f \in \mathbb{R}.$$

Suppose that the input is the real harmonic signal

$$u(t) = u_o \cos(2\pi f_o t), \qquad t \geq 0,$$

with f_o a fixed positive frequency. The general solution of the differential equation corresponding to this input is

$$y(t) = |\hat{h}(f_o)|u_o \cos(2\pi f_o t + \psi_o) + \alpha e^{-t/RC}, \qquad t \geq 0, \tag{1}$$

where

$$|\hat{h}(f_o)| = \frac{1}{\sqrt{4\pi^2 f_o^2 R^2 C^2 + 1}},$$

$$\psi_o = \arg(\hat{h}(f_o)) = -\operatorname{atan}(2\pi f_o RC).$$

The arbitrary constant α in (1) may be found from the initial condition $y(0) = y_o$. It follows that

$$y(t) = |\hat{h}(f_o)|u_o \cos(2\pi f_o t + \psi_o) + [y_o - |h(f_o)|u_o \cos(\psi_o)]e^{-t/RC}, \qquad t \geq 0.$$

The first term is the steady-state response to the real harmonic input, the second the transient response. The responses are shown in Fig. 4.12 for $y_o = 0$.

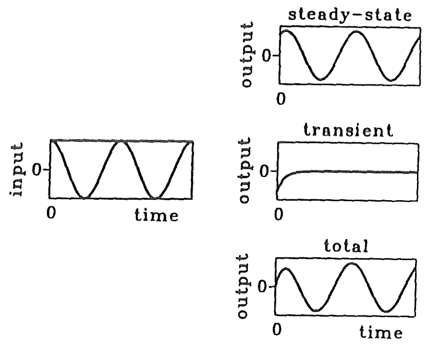


Figure 4.12. Response of the RC network to a real harmonic input. Left: input. Top right: steady-state response. Middle right: transient response. Bottom right: total response. ■

We review the results of this section for sampled linear time-invariant difference systems.

4.7.7. Review: Frequency response function of sampled constant coefficient difference systems. The frequency response function of sampled systems defined on the time axis $\mathbb{Z}(T)$ described by a constant coefficient linear difference equation of the form

$$Q(\sigma^T)y = P(\sigma^T)u$$

exists if and only if the magnitudes of all the poles of the system are strictly less than 1. Its form may easily be established by determining a particular solution of the form $y(t) = y_o e^{j2\pi ft}$, $t \in \mathbb{Z}(T)$, given the harmonic input $u(t) = e^{j2\pi ft}$, $t \in \mathbb{Z}(T)$. It follows that the frequency response function is given by

$$\hat{h}(f) = \frac{P(e^{j2\pi fT})}{Q(e^{j2\pi fT})}, \qquad f \in \mathbb{R}. \qquad ■$$

Frequency Response Functions of Electrical Networks

We conclude this section by describing a simple and elegant way of finding frequency response functions of *electrical networks*. Consider an interconnection of basic electrical network elements such as resistors, capacitors and inductors (see Fig. 4.13.) We assume that the input u to the system originates from a single voltage or current source, and that the output y is a single voltage or current somewhere in the network (see for an example 4.7.8.)

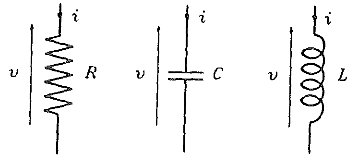


Figure 4.13. Three elementary electrical network elements. Left: resistor. Middle: capacitor. Right: inductor.

From the theory of this section it follows that we can determine the frequency-response function $\hat{h}$ of the network if corresponding to the harmonic input $u(t) = u_o e^{j2\pi ft}$, $t \in \mathbb{R}$, we can find for the output a particular solution $y(t) = y_o e^{j2\pi ft}$, $t \in \mathbb{R}$, with y_o a suitable scalar constant; then $y_o = \hat{h}(f)u_o$.

We exploit the fact that corresponding to the input $u(t) = u_o e^{j2\pi ft}$, $t \in \mathbb{R}$, *every* current and voltage within the network has a particular solution of the form $c\, e^{j2\pi ft}$, $t \in \mathbb{R}$, with c a suitable constant, and first consider each network element individually.

Resistor. The current i through and the voltage v across a resistor with resistance R are related by Ohm's law as $v(t) = Ri(t)$. Suppose that both the current and the voltage are harmonic signals of the form $i(t) = \hat{\imath}\, e^{j2\pi ft}$, $v(t) = \hat{v}\, e^{j2\pi ft}$, $t \in \mathbb{R}$, with $\hat{\imath}$ and $\hat{v}$ constants to be determined. By substitution it follows that $\hat{\imath}$ and $\hat{v}$ are related as $\hat{v} = R\hat{\imath}$, or

$$Z_R(f) = \frac{\hat{v}}{\hat{\imath}} = R.$$

The ratio $Z_R(f) := \hat{v}/\hat{\imath}$ is called the *impedance* of the network element. As the notation indicates, the impedance generally is frequency dependent. The impedance Z_R of the resistor is precisely its resistance R and, hence, is constant.

George Simon Ohm (1787–1854) was a German physicist.

Capacitor. The voltage v across a capacitor and the current i flowing into it are related as $i(t) = C\, dv(t)/dt$, with C its capacitance. Substituting $i(t) = \hat{\imath}\, e^{j2\pi ft}$ and $v(t) = \hat{v}\, e^{j2\pi ft}$, $t \in \mathbb{R}$, it follows that $\hat{\imath} = C\, j2\pi f\hat{v}$, so that the impedance of the capacitor is

$$Z_C(f) = \frac{\hat{v}}{\hat{\imath}} = \frac{1}{j2\pi fC}, \qquad f \neq 0.$$

In terms of the *angular frequency* $\omega = 2\pi f$ the impedance of the capacitor $Z_C(\omega/2\pi) = 1/j\omega C$.

Inductor. The voltage v across an inductor and the current i through i finally, are related by $v(t) = L\, di(t)/dt$, with L its inductance. It easily follows wit $v(t) = \hat{v}\, e^{j2\pi ft}$ and $i(t) = \hat{\imath}\, e^{j2\pi ft}$, $t \in \mathbb{R}$, that $\hat{v} = L\, j2\pi f\hat{\imath}$, and, hence, the impe dance of the inductor is

$$Z_L(f) = j2\pi fL.$$

In terms of the angular frequency it follows that $Z_L(\omega/2\pi) = j\omega L$.

For elementary interconnections of network elements it is very simple to find their *replacement impedances*, that is, the ratio $Z(f) = \hat{v}/\hat{\imath}$ when the voltage across and the current through the interconnection are $\hat{v}\, e^{j2\pi ft}$ and $\hat{\imath}\, e^{j2\pi ft}$, $t \in \mathbb{R}$, respectively. The results are completely analogous to those obtained when analyzing DC networks that only contain resistors.

Consider for instance the series connection of Fig. 4.14 (top), where the individual network elements have impedances Z_1 and Z_2, respectively. It follows that $\hat{v} = \hat{v}_1 + \hat{v}_2 = Z_1\hat{\imath} + Z_2\hat{\imath} = (Z_1 + Z_2)\hat{\imath}$, so that the replacement impedance of the series connection is $Z = Z_1 + Z_2$. For convenience we omit the argument f of the impedances.

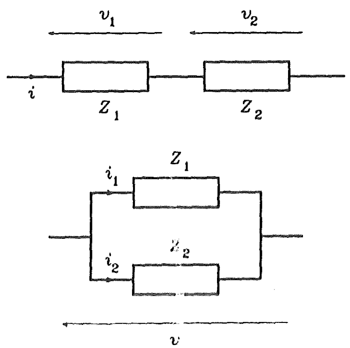


Figure 4.14. Connection of two electrical network elements. Top: series. Bottom: parallel.

For the parallel connection of Fig. 4.14 (bottom), on the other hand, we have $\hat{\imath} = \hat{\imath}_1 + \hat{\imath}_2 = \hat{v}/Z_1 + \hat{v}/Z_2 = (1/Z_1 + 1/Z_2)\hat{v}$. Consequently, the replacement impedance Z of the parallel connection follows from $1/Z = 1/Z_1 + 1/Z_2$.

In this way, given a harmonic input current or voltage of the form $u(t) = \hat{u}\, e^{j2\pi ft}$, $t \in \mathbb{R}$, one may usually easily determine the corresponding output voltage or current $y(t) = \hat{y}\, e^{j2\pi ft}$, $t \in \mathbb{R}$, and thus establish the frequency response function of the network.

4.7.8. Example: RCL network. By way of example we consider the RCL series network, whose diagram is repeated in Fig. 4.15. The input is the voltage produced by the voltage source, and the output is the voltage across the inductance. The first step to find the frequency response function is to construct the replacement network of Fig. 4.16. The impedance Z_1 is the series connection of the resistor (with impedance R) and the capacitor (with impedance $1/j\omega C$), so that $Z_1 = R + 1/j\omega C$. The impedance Z_2 is that of the inductor, so that $Z_2 = j\omega L$. We use the angular frequency ω throughout this example.

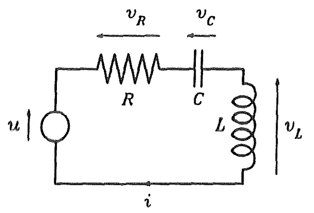


Figure 4.15. RCL network.

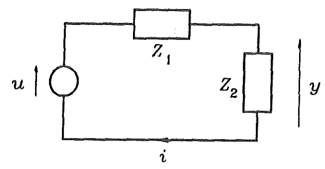


Figure 4.16. Replacement network.

Given the input voltage $u(t) = \hat{u}\, e^{j\omega t}$, $t \in \mathbb{R}$, the current through the network is $i(t) = \hat{\imath}\, e^{j\omega t}$, $t \in \mathbb{R}$, where

$$\hat{\imath} = \frac{\hat{u}}{Z_1 + Z_2}.$$

It follows that the voltage across the impedance Z_2 is $y(t) = \hat{y}\, e^{j\omega t}$, $t \in \mathbb{R}$, where

$$\hat{y} = Z_2\hat{\imath} = \frac{Z_2}{Z_1 + Z_2}\hat{u}.$$

Thus, the system frequency response function is

$$\hat{h}(f) = \hat{h}(\omega/2\pi) = \frac{\hat{y}}{\hat{u}} = \frac{Z_2}{Z_1 + Z_2} = \frac{j\omega L}{R + \dfrac{1}{j\omega C} + j\omega L}$$

$$= \frac{(j\omega)^2}{(j\omega)^2 + \dfrac{R}{L}j\omega + \dfrac{1}{LC}}.$$

This agrees with what we found in Example 4.7.3(b).

4.8 PROBLEMS

'n the first problem we follow up Example 3.2.13 as an instance of a nonlinear differential system.

4.8.1. Response of the car. As found in Example 3.2.13, the speed of a car may be described by the differential equation

$$M\frac{dv(t)}{dt} = cu(t) - Bv^2(t), \qquad t \in [0, \infty).$$

Here v is the car speed, u (ranging between 0 and 1) the throttle position, and M, c and B physical constants. If the throttle position has the constant value 1, the speed has a corresponding stationary value $v_{\max}$ that satisfies $0 = c - Bv_{\max}^2$ and, hence, is given by $v_{\max} = \sqrt{c/B}$.

(a) Define

$$w = \frac{v}{v_{\max}}$$

as the speed expressed as fraction of the top speed, and show that w satisfies the differential equation

$$\frac{dw(t)}{dt} = \alpha[u(t) - w^2(t)], \qquad t \geq 0,$$

where $\alpha = \sqrt{Bc}/M$.

(b) Take the initial time equal to 0, assume that at this time the car has speed w_o (as fraction of the top speed) and that the throttle position remains constant at the value u_o. Use separation of variables to show that the solution of the differential equation is

$$w(t) = w_o\frac{1 + \dfrac{w_\infty}{w_o}\tanh(\alpha w_\infty t)}{1 + \dfrac{w_o}{w_\infty}\tanh(\alpha w_\infty t)}, \qquad t \geq 0,$$

where $w_\infty = \sqrt{u_o}$ is the stationary speed corresponding to the constant throttle setting u_o, and tanh is the hyperbolic tangent given by

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad -\infty < x < \infty.$$

(c) Plot the response of the speed of the car w if $\alpha = 1/10$ [s^{-1}], $w_o = 0.5$ and $w_\infty = 0.6$.

(d) What is the behavior of the speed w if $w_o = 0$ and $u_o = 1$?

The next problems deal with linear difference and differential systems as introduced in Section 4.3.

4.8.2. Various difference systems. Show that the following discrete-time systems may be described by a constant coefficient linear difference equation of the form

$$Q(\sigma)y = P(\sigma)u,$$

and determine the polynomials Q and P.

(a) The *delay* system described by

$$y(n) = u(n - M), \qquad n \in \mathbb{Z},$$

with M a nonnegative integer.

(b) The *tapped delay-line*, which is a system described by an expression of the form

$$y(n) = a_M u(n) + a_{M-1}u(n - 1) + \cdots + a_0 u(n - M), \qquad n \in \mathbb{Z},$$

with $a_0, a_1, \cdots, a_M$ real coefficients, and M a nonnegative integer. The reason for the name is that the system may be implemented with a delay line as shown in Fig. 4.17. The system is also known as a *moving averager* (MA system), or *transversal filter*.

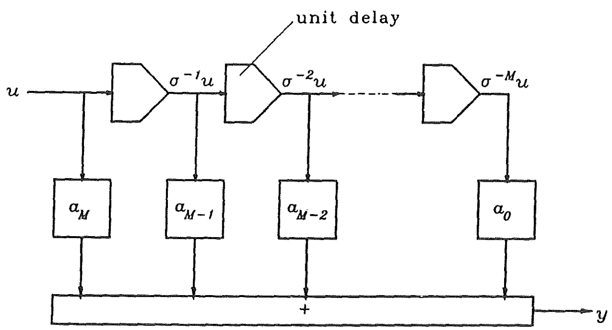


Figure 4.17. A tapped delay line.

(c) An *auto-regressive* (AR) system, which is a system described by an expression of the form

$$y(n) = b_{N-1}y(n - 1) + b_{N-2}y(n - 2) + \cdots + b_0 y(n - N) + a_0 u(n),$$

$n \in \mathbb{Z}$, with $a_0, b_0, b_1, \cdots, b_{N-1}$ real coefficients, and N a nonnegative integer. The reason for the name is that the current value of the output *regresses* on the previous values.

(d) An *ARMA* (auto-regressive moving-average) system, which is a system whose output is given by an expression of the form

$$y(n) = a_M u(n) + a_{M-1}u(n - 1) + \cdots + a_0 u(n - M)$$
$$+ b_{N-1}y(n - 1) + b_{N-2}y(n - 2) + \cdots + b_0 y(n - N),$$

$n \in \mathbb{Z}$, where $a_0, a_1, \cdots, a_M, b_0, b_1, \cdots, b_{N-1}$ are real coefficients and M and N nonnegative integers. An ARMA system is a combination of an auto-regressive and a moving-average system.

4.8.3. Some differential systems. For the following systems, determine the differential equation that relates the continuous-time input u and output y. If the system is linear and time-invariant bring the equation in the form $Q(D)y = P(D)u$ and exhibit the polynomials P and Q.

(a) *RL network*. The input to the network of Fig. 4.18 is the voltage u of the voltage source, while the output of the system is the voltage y across the resistor.

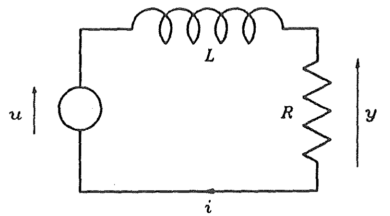


Figure 4.18. An RL circuit.

(b) *Double spring-mass system*. In the double spring-mass system of Fig. 4.19, the springs have spring constants k_1 and k_2, while the two blocks have masses m_1 and m_2, respectively. The blocks move in one dimension only. The input to the system is the displacement u of the left-hand side of the leftmost spring, which is measured relative to a fixed point. The displacements of the blocks are z_1 and z_2, respectively, both taken relative to fixed points that correspond to the positions of the blocks when the system is at rest. There is no friction. The output y of the system is the position z_2 of the second block. *Hints:* Argue that the system satisfies the equations

$$m_1\ddot{z}_1 = -k_1(z_1 - u) + k_2(z_2 - z_1),$$
$$m_2\ddot{z}_2 = \qquad\qquad - k_2(z_2 - z_1).$$

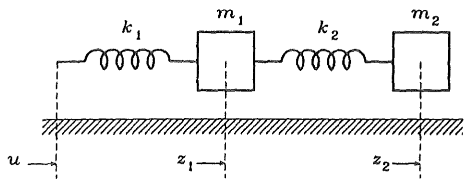


Figure 4.19. A double spring-mass system.

Differentiate the second equation twice and eliminate z_1 and $\ddot{z}_1$ from the three equations that now have been obtained. This leads to a differential equation of order 4.

Linearity, time-invariance, and initially-at-rest-systems are introduced in Section 4.3.

4.8.4. Linearity and time-invariance. Determine whether the following differential systems are (i) linear, (ii) time-invariant.

(a) $$t\frac{dy(t)}{dt} + y(t) = u(t), \qquad t \in \mathbb{R}.$$

(b) $$\frac{dy(t)}{dt} + y(t) = [u(t)]^2, \qquad t \in \mathbb{R}.$$

(c) $$\frac{dy(t)}{dt} + y(t) = t[u(t)]^2, \qquad t \in \mathbb{R}.$$

(d) $$\left[\frac{dy(t)}{dt}\right]^2 = [u(t)]^2, \qquad t \in \mathbb{R}.$$

4.8.5. Initially-at-rest systems. Determine the initially-at-rest response of each of the following systems by first assuming the system to be at rest at some initial time and next letting the initial time approach $-\infty$. What is the impulse response h of the initially-at-rest system?

(a) The savings account of Example 4.2.3(b).
(b) The delay line of Problem 4.8.2(a).
(c) The tapped delay line of Problem 4.8.2(b).
(d) The continuous-time differential system

$$\frac{dy(t)}{dt} = u(t), \qquad t \in \mathbb{R}.$$

In the next series of problems we deal with the response of linear time-invariant difference and differential systems, as discussed in Section 4.4.

4.8.6. Basis solutions. Determine the basis solutions, both in complex and in real form, of the linear constant coefficient difference and differential equations describing the following systems.

(a) The auto-regressive system

$$y(n+2) - \frac{1}{4}y(n) = u(n+2), \qquad n \in \mathbb{T},$$

both when $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{Z}_+$.

(b) The auto-regressive system

$$y(n+2) - y(n+1) + \frac{5}{16}y(n) = u(n+2), \qquad n \in \mathbb{T},$$

both when $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{Z}_+$.

(c) The tapped delay line (or moving averager)

$$y(n+M) = a_0u(n) + a_1u(n+1) + \cdots + a_Mu(n+M), \qquad n \in \mathbb{T},$$

both when $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{Z}_+$.

(d) The ARMA system

$$y(n+2) - \frac{1}{4}y(n) = u(n+2) + u(n+1), \qquad n \in \mathbb{T},$$

both when $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{Z}_+$.

(e) The LR network of Problem 4.8.3(a), which is described by the differential equation

$$\frac{L}{R}\frac{dy(t)}{dt} + y(t) = u(t), \qquad t \in \mathbb{R}.$$

(f) The double spring-mass system of Problem 4.8.3(b), which is described by the differential equation

$$y^{(4)}(t) + \left(\omega_1^2 + \omega_2^2 + \frac{m_2}{m_1}\omega_2^2\right)y^{(2)}(t) + \omega_1^2\omega_2^2\, y(t) = \omega_1^2\omega_2^2\, u(t), \qquad t \in \mathbb{R},$$

where $\omega_1^2 = k_1/m_1$ and $\omega_2^2 = k_2/m_2$. Assume that

$$\omega_1^2 = 3/2, \quad \omega_2^2 = 4/3, \quad m_2/m_1 = 1/8.$$

(g) The "$n$fold integrator," which is the system described by the differential equation

$$y^{(n)}(t) = u(t), \qquad t \in \mathbb{R}.$$

(h) The system described by the differential equation

$$y''(t) - y(t) = u'(t) - u(t), \qquad t \in \mathbb{R}.$$

4.8.7. Initial value problems. Solve the following initial value problems.

(a) *Fibonacci numbers.* The Fibonacci numbers are the solution of the difference equation

$$y(n+2) = y(n+1) + y(n), \qquad n = 0, 1, 2, \cdots,$$

with $y(0) = 0$ and $y(1) = 1$.

> Fibonacci was the nickname of Leonardo of Pisa (ca. 1180–ca. 1250). Born of the trading class, Fibonacci learned Arabic mathematics on his travels and wrote about it in his book *Liber Abaci* (1202).

(b) *ARMA system:*

$$y(n+2) - \frac{1}{4}y(n) = u(n+2) + u(n+1), \qquad n = 0, 1, 2, \cdots,$$

with $u(n) = 1$ for $n \geq 0$, and $y(0) = y(1) = 1$.

(c) *LR network:*

$$\frac{L}{R}\frac{dy(t)}{dt} + y(t) = u(t), \qquad t \geq 0,$$

with $u(t) = t$ for $t \geq 0$ and $y(0) = 0$.

(d) *Double spring-mass system:*

$$y^{(4)}(t) + \left(\omega_1^2 + \omega_2^2 + \frac{m_2}{m_1}\omega_2^2\right)y^{(2)}(t) + \omega_1^2\omega_2^2\, y(t) = \omega_1^2\omega_2^2\, u(t), \qquad t \geq 0,$$

where $\omega_1^2 = 3/2$, $\omega_2^2 = 4/3$, $m_2/m_1 = 1/8$, with $u(t) = 0$ for $t \geq 0$, and $y(0) = 1$, $y^{(1)}(0) = y^{(2)}(0) = y^{(3)}(0) = 0$.

(e) *Multiple integrator:*

$$y^{(n)}(t) = u(t), \qquad t \geq 0,$$

with $u(t) = t^k$, $t \geq 0$, and $y^{(i)}(0) = 0$ for $i = 0, 1, \cdots, n-1$. Both n and k are nonnegative integers.

Initially-at-rest linear time-invariant difference and differential systems are convolution systems. In Section 4.5 it is shown how their impulse response may be found.

4.8.8. Impulse and step responses. Determine the impulse and step responses of each of the following systems. Sketch them. Establish for each system whether it is non-anticipating.

(a) The auto-regressive system

$$y(n+2) - \frac{1}{4}y(n) = u(n+2), \qquad n \in \mathbb{Z}.$$

(b) The auto-regressive system

$$y(n+2) - y(n+1) + \frac{5}{16}y(n) = u(n+2), \qquad n \in \mathbb{Z}.$$

(c) The tapped delay line (or moving averager)

$$y(n+M) = a_0u(n) + a_1u(n+1) + \cdots + a_Mu(n+M), \qquad n \in \mathbb{Z}.$$

(d) The ARMA system

$$y(n+2) - \frac{1}{4}y(n) = u(n+2) + u(n+1), \qquad n \in \mathbb{Z}.$$

(e) The LR network of Problem 4.8.3(a), which is described by the differential equation

$$\frac{L}{R}\frac{dy(t)}{dt} + y(t) = u(t), \qquad t \in \mathbb{R}.$$

(f) The double spring-mass system of Problem 4.8.3(b), which is described by the differential equation

$$y^{(4)}(t) + \left(\omega_1^2 + \omega_2^2 + \frac{m_2}{m_1}\omega_2^2\right)y^{(2)}(t) + \omega_1^2\omega_2^2\, y(t) = \omega_1^2\omega_2^2\, u(t), \qquad t \in \mathbb{R},$$

where $\omega_1^2 = k_1/m_1$ and $\omega_2^2 = k_2/m_2$. Assume that $\omega_1^2 = 3/2$, $\omega_2^2 = 4/3$, $m_2/m_1 = 1/8$.

(g) The "$n$fold integrator," which is the system described by the differential equation

$$y^{(n)}(t) = u(t), \qquad t \in \mathbb{R}.$$

(h) The system described by the differential equation

$$y''(t) - y(t) = u'(t) - u(t), \qquad t \in \mathbb{R}.$$

The next problem deals with BIBO and CICO stability of difference and differential systems as discussed in Section 4.6.

4.8.9. BIBO and CICO stability. For the following difference and differential systems, investigate whether (i) the initially-at-rest system is BIBO stable, (ii) the IO system described by the difference or differential equation is BIBO stable, and (iii) the IO system is CICO stable.

(a) The delay of Problem 4.8.2(a).
(b) The "Fibonacci system" of 4.8.7(a).
(c) The auto-regressive system of Problem 4.8.8(a).

(d) The auto-regressive system of Problem 4.8.8(b).
(e) The tapped delay line of Problem 4.8.8(c).
(f) The ARMA system of Problem 4.8.8(d).
(g) The LR network of Problem 4.8.8(e), with L and R both positive.
(h) The RCL network of Example 4.2.4(c), with R, C and L all positive.
(i) The double spring-mass system of Problem 4.8.8(f).
(j) The $n$fold integrator of Problem 4.8.8(g).
(k) The second-order system of Problem 4.8.8(h).

The frequency response of difference and differential systems is treated in Section 4.7.

4.8.10. Frequency response of difference and differential systems. Determine the characteristic roots, the poles and the zeros, if any, of the following systems. Establish whether each system has a frequency response function, and determine it if it exists.

(a) The delay of Problem 4.8.2(a).
(b) The auto-regressive system of Problem 4.8.8(a).
(c) The auto-regressive system of Problem 4.8.8(b).
(d) The tapped delay line of Problem 4.8.8(c), with $M = 2$, $a_0 = 4$, $a_1 = -4$, $a_2 = 1$.
(e) The ARMA system of Problem 4.8.8(d).
(f) The LR network of Problem 4.8.8(e).
(g) The double spring-mass system of Problem 4.8.8(f).
(h) The $n$fold integrator of Problem 4.8.8(g).

4.8.11. Steady-state and transient response to harmonic inputs.

(a) *Exponential smoother.* Consider the exponential smoother $y(n + 1) = ay(n) + (1 - a)u(n + 1)$, $n \in \mathbb{Z}_+$, with $a = \sqrt{3}/3$. Determine the steady-state and transient response of the smoother to the real harmonic input

$$u(n) = \cos(2\pi fn), \qquad n \in \mathbb{Z}_+,$$

with the initial condition $y(0) = 0$, for $f = 1/4$.

(b) *Second-order differential system.* Determine the steady-state and transient response of the system $y'' + 2y' + y = u' - u$ to the real harmonic input

$$u(t) = \cos(2\pi ft), \qquad t \geq 0,$$

with the initial conditions $y(0) = y'(0) = 0$, where $2\pi f = 1$.

4.8.12. Frequency response functions of electrical networks. Determine the frequency response functions of the following electrical networks.

(a) The second-order RC network of Fig. 4.20, whose input is the current u produced by the current source, and whose output y is the current through the capacitor C_2.
(b) The second-order LC network of Fig. 4.21, whose input is the voltage u of the voltage source, and whose output y is the voltage across the capacitor C_2.

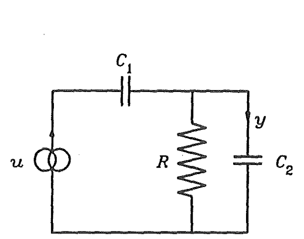


Figure 4.20. An RC network.

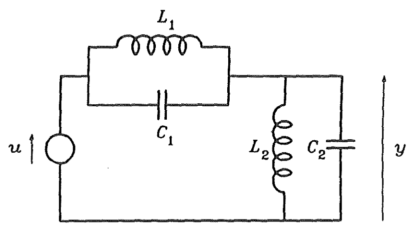


Figure 4.21. An LC network.

4.9 COMPUTER EXERCISES

The computer exercises for Chapter 4 deal with the numerical solution of difference and differential equations, computing characteristic roots, poles and zeros, and Bode plots.

4.9.1. Solution of difference equations.

(a) *Fibonacci equation.* The Fibonacci equation is the second-order difference equation

$$y(n + 2) = y(n) + y(n + 1), \qquad n = 0, 1, 2, \cdots.$$

The *Fibonacci numbers* are its solution for the initial conditions $y(0) = 0$, $y(1) = 1$. Solve the Fibonacci equation numerically for $0 \leq n \leq 20$ with the given initial conditions, and plot the solution y. What is $y(20)$? *Hint:* The equation may be converted into a *pair* of first-order difference equations, which are easier to program, by defining $x_1(n) = y(n - 1)$ and $x_2(n) = y(n)$ for $n = 1, 2, \cdots$. Determine difference equations for x_1 and x_2 and program these.

(b) *Exponential smoother.* The exponential smoother is described by the first-order difference equation

$$y(n + 1) = ay(n) + (1 - a)u(n + 1), \qquad n = 0, 1, 2, \cdots.$$

(b.1) Write a macro or simple program to solve the difference equation numerically on the time axis $\{0, 1, 2, \cdots, 100\}$, for a given input signal u and given initial condition $y(0) = 0$, with the constant a externally defined.
(b.2) Let $a = 0.5$, and compute and plot the response of the exponential smoother to a pure noise input. Amplitude scale the response such that its root mean square (rms) value is 1, and use the resulting signal as the input for (b.3). *Hint:* In SIGSYS, generate the noise input as the standard signal `noiseplus`.

(b.3) Use the signal computed in (b.2) as input to the exponential smoother for $a = 0.2$, $a = 0.5$ and $a = 0.9$. Plot the corresponding outputs and observe the different smoothing effects.

Numerical solution of differential equations. The numerical solution of a set of differential equations is often referred to as the numerical *integration* of the equations. Numerical integration involves a *step size* along the time axis. In integration routines where the step size is not automatically selected or adjusted a practical approach to determine the step size is the following. First choose the step size tentatively as, say, 1/100 of the time interval over which the differential equations are to be integrated. If the solution becomes unbounded (which often happens if the step size is too large, and is manifested by numerical overflow), then reduce the step size to, say, half the size. After a solution has been obtained, recompute with a smaller (again, say, halved) step size. If reducing the step size produces no significant change, the step size is small enough. An alternative to reducing the step size is switching to a more powerful integration routine, say from Runge-Kutta 2 to Runge-Kutta 4.

4.9.2. Numerical solution of differential equations. Differential equations of the form

$$F\left[y(t), \frac{dy(t)}{dt}, \cdots, \frac{d^N y(t)}{dt^N}, t\right] = 0, \qquad t \geq 0,$$

most conveniently may be solved numerically by converting them to a *set* of N *first-order* differential equations. First, assume that for each t the given equation may uniquely be solved for $d^N y(t)/dt^N$ in the form

$$\frac{d^N y(t)}{dt^N} = G\left[y(t), \frac{dy(t)}{dt}, \cdots, \frac{d^{N-1} y(t)}{dt^{N-1}}, t\right], \qquad t \geq 0.$$

Next, define the N auxiliary signals

$$\begin{aligned} x_1(t) &= y(t), \\ x_2(t) &= y^{(1)}(t), \\ \cdots & \;\; \cdots \\ x_N(t) &= y^{(N-1)}(t), \end{aligned}$$

all for $t \geq 0$. It is easily seen that $x_1, x_2, \cdots, x_N$ satisfy the set of first-order simultaneous differential equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t), \\ \cdots & \;\; \cdots \\ \dot{x}_{N-1}(t) &= x_N(t), \\ \dot{x}_N(t) &= G[x_1(t), x_2(t), \cdots, x_N(t), t], \end{aligned}$$

all for $t \geq 0$. The initial conditions for this set of differential equations follow directly from the given initial conditions $y(0)$, $y^{(1)}(0)$, $\cdots$, $y^{(N-1)}(0)$ for the original differential equation. For the numerical solution of sets of first-order differential equations many methods exist (see Section 5.4), and a plethora of standard software is available.

(a) *RC network.* The RC network (with the voltage across the capacitor as output), is described by the differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = u(t), \qquad t \geq 0.$$

Take $RC = 1$, and solve the differential equation numerically on the interval [0, 20] for the input

$$u(t) = \cos(2\pi f_o t), \qquad t \geq 0,$$

with $2\pi f_o = 1$ and the initial condition $y(0) = 0$.

(b) *Moving car* (compare 4.8.1.) The moving car is described by the normalized differential equation

$$\frac{dw(t)}{dt} = \alpha[u(t) - w^2(t)], \qquad t \geq 0,$$

with w the speed of the car as fraction of the top speed, and u the throttle position. Integrate the differential equation numerically on the interval [0, 100] for the input

$$u(t) = 0.36, \qquad t \geq 0,$$

with the initial condition $w(0) = 0.5$, while $\alpha = 0.1$ [s^{-1}]. Compare the solution numerically with the analytical result of 4.8.1.

(c) *RCL network.* The RCL circuit of 4.2.4(c) with the current as output is described by the differential equation

$$y^{(2)}(t) + \frac{R}{L} y^{(1)}(t) + \frac{1}{LC} y(t) = \frac{1}{L} u^{(1)}(t), \qquad t \geq 0.$$

Suppose that the input u is identical to zero, and convert the differential equation to a set of two first-order equations. Choose $R = 1/4$, $C = 1$ and $L = 1$ and integrate the differential equations over the interval [0, 20] for the initial conditions $y(0) = 1$, $y^{(1)}(0) = 0$.

4.9.3. Basis solutions of constant coefficient linear difference and differential equations. (Compare 4.8.6.) Compute the characteristic roots of the linear constant coefficient difference and differential equations that describe the following systems. Determine and plot the basis solutions of the homogeneous equations.

(a) The auto-regressive system

$$y(n+2) - y(n+1) + \frac{5}{16}y(n) = u(n+2), \qquad n \in \mathbb{Z}_+.$$

(b) The double spring-mass system, described by the differential equation

$$y^{(4)}(t) + \left(\omega_1^2 + \omega_2^2 + \frac{m_2}{m_1}\omega_2^2\right)y^{(2)}(t) + \omega_1^2\omega_2^2\, y(t) = \omega_1^2\omega_2^2\, u(t), \qquad t \in \mathbb{R}_+,$$

with $\omega_1^2 = 3/2$, $\omega_2^2 = 4/3$ and $m_2/m_1 = 1/8$.

(c) The discrete-time "echo" system, described by the difference equation

$$y(n) = u(n) + \alpha y(n-N), \qquad n = N, N+1, N+2, \cdots,$$

with N a nonnegative integer and α a real constant. Let $N = 4$ and $\alpha = 7/8$.

4.9.4. Stability of constant coefficient linear difference and differential systems. Compute the characteristic values, poles and zeros of the following linear time-invariant difference and differential systems. Determine the BIBO and CICO stability of each system.

(a) The difference system $Q(\sigma)y = P(\sigma)u$, where P and Q are the polynomials

$$Q(\lambda) = 1 + 2\lambda^4, \qquad P(\lambda) = 1 + \lambda.$$

(b) The differential system $Q(D)y = P(D)u$, where the polynomials P and Q are given as in (a).

(c) The difference system $Q(\sigma)y = P(\sigma)u$, where P and Q are the polynomials

$$Q(\lambda) = 1 - \lambda^4, \qquad P(\lambda) = 1 - \lambda + \lambda^2 - \lambda^3.$$

(d) The differential system $Q(D)y = P(D)u$, where the polynomials P and Q are given as in (c).

4.9.5. Asympotic Bode plots. *Bode plots* graphically represent frequency response functions of continuous-time systems as follows:

(i) the logarithm of the magnitude is plotted versus the logarithm of frequency, and
(ii) the phase is plotted linearly versus the logarithm of frequency.

This often is a convenient way of displaying frequency response functions.

Suppose that the frequency response function $\hat{h}$ is the rational function given by

$$\hat{h}(\omega/2\pi) = \frac{P(j\omega)}{Q(j\omega)}, \qquad \omega \in \mathbb{R},$$

with P and Q polynomials with real coefficients. Because $\hat{h}$ is conjugate symmetric it is sufficient to plot it for positive angular frequencies only.

According to the fundamental theorem of algebra, the polynomials P and Q may be factored as

$$P(\lambda) = p_0 \prod_k (\lambda - \zeta_k), \qquad Q(\lambda) = q_0 \prod_k (\lambda - \pi_k),$$

where the ζ_k are the roots of P, π_k those of Q, and p_0 and q_0 constants. It follows that

$$\log|\hat{h}(\omega/2\pi)| = \log(|p_0/q_0|) + \sum_k \log|j\omega - \zeta_i| - \sum_k \log|j\omega - \pi_k|. \tag{1}$$

Let us look at one term individually, for instance $\log|j\omega - \alpha|$, with α a complex number. We have

$$\log|j\omega - \alpha| \approx \begin{cases} \log|\alpha| & \text{for } \omega << |\alpha|, \\ \log(\omega) & \text{for } \omega >> |\alpha|, \end{cases} \qquad \omega \in \mathbb{R}_+.$$

This shows that on a logarithmic frequency scale for *low* frequencies the term $\log|j\omega - \alpha|$ has a constant as asymptote, while for *high* frequencies it has a straight line $\log(\omega)$ with slope 1 as asymptote. Figure 4.22 shows the position of the asymptotes. The asymptotes *intersect* at the angular frequency $\omega = |\alpha|$, which is called the *break frequency* corresponding to the factor $j\omega - \alpha$. Near the break frequency the plot of $|j\omega - \alpha|$ deviates most significantly from the asymptotes.

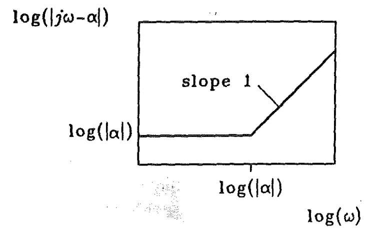


Figure 4.22. Asymptotes of the magnitude of $j\omega - \alpha$ in a double-logarithmic plot.

The asymptotes of the various terms in (1) may be used to obtain a quick sketch of the general behavior of the Bode magnitude plot. Consider for instance the frequency response function

$$\hat{h}(\omega/2\pi) = \frac{P(j\omega)}{Q(j\omega)},$$

where

$$P(\lambda) = 4(\lambda + 1),$$
$$Q(\lambda) = \lambda^2 + \lambda + 4.$$

Bode plots.

Bode plots graphically represent frequency response functions of continuous-time systems as follows

(i) the logarithm of the magnitude is plotted versus the logarithm of frequency, and
(ii) the phase is plotted linearly versus the logarithm of frequency.

This often is a convenient way of displaying frequency response functions.

A Bode plot of a frequency response function $\hat{h}(\omega/2\pi)$, $\omega \in \mathbb{R}_+$, may be prepared as follows. We assume that $\hat{h}$ is specified in analytic form. First determine the frequency interval over which the plot is made. A good way of doing this is by first sketching an asymptotic magnitude plot as explained in 4.9.5. The frequency interval may then be chosen, say, starting from one or two frequency decades down from the lower break frequency of interest and extending to one or two decades up from the highest break frequency. (A *decade* is a factor of 10.)

Suppose that it is decided to plot over the angular frequency range extending from 10^a to 10^b. To plot the angular frequency logarithmically, first define an underlying linear signal axis with domain $[a, b]$ and a sufficient number of points to make nice plots (say 100). Next, compute an exponentially sampled angular frequency axis by evaluating 10^r with r ranging over the underlying linear axis. Finally, evaluate the frequency response function $\hat{h}$ on the angular frequency axis, and from this obtain the $\log_{10}$ of its magnitude and its phase.

In SIGSYS, the underlying linear axis could be created as `LOGOM = a:(b-a)/100:b`. Next, the exponentially spaced angular frequency axis may be obtained as `OM = 10^LOGOM`. Suppose that the frequency response function to be evaluated is given by $\hat{h}(\omega/2\pi) = P(j\omega)/Q(j\omega)$, with P and Q given polynomials. Then we may evaluate $\hat{h}$ on the angular frequency axis as `hhat = P(j*OM)/Q(j*OM)`. The logarithm of the magnitude and phase follow as `ahhat = log10(abs(hhat))` and `phhat = arg(hhat)`, respectively.

Hendrik Wade Bode (1905–1982) was an American research engineer who did important work on network design during his long association with the Bell Telephone Laboratories.

The numerator polynomial P has a single root -1, whose magnitude is 1. The de nominator Q has a complex pair of roots, each of which has magnitude $\sqrt{4} = 2$. A low frequencies $\hat{h}$ has the low frequency asymptote $\hat{h}(0) = P(0)/Q(0) = 1$. As fre quency increases, the first break frequency that is encountered is 1. Because thi break frequency is contributed by the numerator, at this point an asymptote witl slope 1 sets in, as indicated in Fig. 4.23. The next break frequency 2 represents a double root contributed by the denominator. As a consequence, an asymptote witl slope -2 sets in, which results in a slope change of the asymptotic magnitude plot from 1 to -1. Figure 4.23 shows the resulting asymptotic plot.

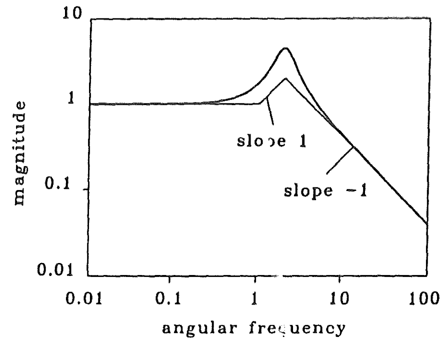


Figure 4.23. Asymptotic and actual Bode magnitude plots.

The figure also gives the full Bode magnitude plot. The agreement with the asymptotic plot is very good at high and low frequencies, but less so near the break frequencies.

(a) Reproduce Fig. 4.23 by plotting the Bode magnitude plot and the asymptotic plot of the given frequency response function in one frame.
(b) Compute the break frequencies, sketch the asymptotic Bode magnitude plot, and display the Bode magnitude and phase plots of the frequency response function

$$\hat{h}_c(\omega/2\pi) = \frac{\frac{1}{LC}}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}}, \qquad \omega \in \mathbb{R}_+,$$

with $R = 1/4$, $C = 1$ and $L = 1$. This is the frequency response function of the RCL network of 4.2.4(c) when the output is the voltage across the capacitor.

(c) Repeat (b) for the frequency response function

$$\hat{h}_R(\omega/2\pi) = \frac{\frac{R}{L}j\omega}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}}, \qquad \omega \in \mathbb{R}_+,$$

with the numerical values of (b).

(d) Repeat (b) for the frequency response function

$$\hat{h}_L(\omega/2\pi) = \frac{(j\omega)^2}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}}, \qquad \omega \in \mathbb{R}_+,$$

with the numerical values of (b).

4.9.6. Butterworth filters. A normalized *Butterworth filter* of order N is a continuous-time differential system described by a differential equation of the form

$$Q_N(D)y = u,$$

where the polynomial Q_N is obtained as follows. The polynomial

$$\phi_N(\lambda) = \lambda^{2N} + (-1)^N,$$

with N a positive integer, has $2N$ roots λ_i, $i = 1, 2, \cdots, 2N$, that may be arranged such that λ_i, $i = 1, 2, \cdots, N$, all have strictly negative real parts, while the remaining N roots all have strictly positive real parts. Then

$$Q_N(\lambda) = \prod_{i=1}^{N} (\lambda - \lambda_i).$$

The frequency response function of the Nth order Butterworth filter is

$$\hat{h}_N(\omega/2\pi) = \frac{1}{Q_N(j\omega)}, \qquad \omega \in \mathbb{R}_+.$$

Butterworth filters are low-pass filters. The angular cut-off frequency of a normalized Butterworth filter is 1. The cut-off frequency may be shifted to ω_o by modifying the frequency response function to

$$\frac{1}{Q_N(j\omega/\omega_o)}, \qquad \omega \in \mathbb{R}_+,$$

which is the frequency response function of the differential system

$$Q(D/\omega_o)y = u.$$

(a) Find the Butterworth polynomials Q_N for $N = 1, 2, 3, 4$, and 5.

(b) Make Bode plots of the magnitude and phase of the normalized Butterworth frequency response functions $\hat{h}_N$ as a function of angular frequency for $N = 1, 2, 3, 4$, and 5. Observe how the low-pass nature of the frequency response function improves as N increases.

Butterworth filters date back to the early days of electronics: S. Butterworth, "On the theory of filter amplifiers," *Wireless Engineer*, London, vol. 7, 1930, pp. 536–541.

5

State Description of Systems

5.1 INTRODUCTION

In Chapters 3 and 4 we considered systems described by their input-output relationship. Such a description reduces the system to a "black box," whose internal mechanisms are ignored. In the present chapter we study the *state description* of systems, which encompasses not only their external but also their internal behavior. Such a description may often be obtained by direct application of the physical or other fundamental laws that govern the system. The state description is of eminent importance for understanding, analyzing and simulating systems.

In Section 5.2 state systems are first informally and then formally introduced. By a progression of assumptions this section leads to linear time-invariant discrete- and continuous-time systems described by *state difference* and *state differential* equations, respectively, together with an *output equation*. These systems form the focus for the remainder of the chapter. In Section 5.3 we show how discrete-time difference systems and continuous-time differential systems may be represented as state difference systems and state differential systems, respectively.

In Section 5.4 the existence of solutions of state difference and differential equations is discussed and methods are presented for their numerical solution. For linear systems the solution of state difference and differential equations is given in Section 5.5 using the state transition matrix.

Section 5.6 presents the details of the *modal* analysis of linear time-invari[ant] state systems, based on the diagonal form of the system matrix. This section leads [to] a considerable improvement of the understanding of the internal behavior of lin[ear] time-invariant state difference and differential systems. Section 5.7 is devoted to t[he] definition and analysis of the *stability* of state systems. In Section 5.8, finally, t[he] frequency response of linear time-invariant state systems is analyzed.

> The idea of state, though not under this name, has been around in physics for some time. What is called "phase" in statistical mechanics, for instance, is precisely the notion state. In system theory the state description broke through in the late 1950s.

5.2 STATE DESCRIPTION OF SYSTEMS

In this section we first introduce the idea of the *state* of a system, next present a few examples, and then proceed to a formal definition of *input-output-state* systems.

The Notion of State

Input-output systems are described by the *input-output rule,* which amounts to a specification of all possible input-output pairs (u, y). Such a description is adequate for many purposes, but suffers from the following defect: Suppose that we are interested in the response $y(t)$, $t \geq t_o$, from some fixed time t_o on, given the input $u(t)$, $t \geq t_o$, from that same time on. It then is necessary to know the *entire past input* $u(t)$, $t < t_o$, as well. In many instances, this information is much more than is required to determine the output from time t_o on. What is really needed is to know the "state" in which the system is. The state of the system summarizes the past of the system insofar as relevant for its future behavior.

What the "state" of the system is may often be ascertained by physical or other fundamental considerations. For electrical circuits, for instance, we know from physics that if the charges of the capacitors and the magnetic fluxes contained by the inductors in the network are known at some fixed time we may predict the future behavior of the system without any information about past values of the network variables. The charges of the capacitors together with the fluxes contained by the inductors constitute the *state* of the network. The electromagnetic laws that apply to the network determine the *evolution* of the state. We illustrate this by a simple example.

5.2.1. Example: RCL network. Consider the RCL network of Example 4.2.4(c), whose diagram is repeated in Fig. 5.1. Rather than obtaining a single differential equation of order two that relates input and output, as in Example 4.2.4(c), we look for a set of *first-order* differential equations that determine the rate of

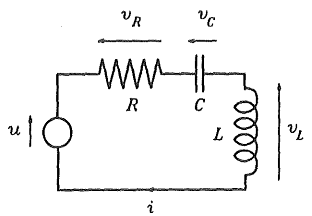


Figure 5.1. An RCL network.

change of the charge q of the capacitor and the flux ϕ contained by the inductor. These two quantities determine the energy that is stored in the network and hence together constitute its state. By Kirchhoff's current law we have

$$\frac{dq(t)}{dt} = i(t),$$

with i the current through the circuit. Since $i(t) = \phi(t)/L$ this may be rewritten as

$$\frac{dq(t)}{dt} = \frac{1}{L}\phi(t). \tag{1}$$

On the other hand we have for the inductor

$$\frac{d\phi(t)}{dt} = v_L(t),$$

with v_L the voltage across the inductor. By Kirchhoff's voltage law it follows that

$$u(t) = v_R(t) + v_C(t) + v_L(t),$$

with v_R the voltage across the resistor and v_C that across the capacitor. Because $v_R = Ri = R\phi/L$ and $v_C = q/C$ this may be rewritten as

$$v_L(t) = u(t) - \frac{R}{L}\phi(t) - \frac{1}{C}q(t),$$

so that

$$\frac{d\phi(t)}{dt} = u(t) - \frac{R}{L}\phi(t) - \frac{1}{C}q(t). \tag{2}$$

This equation together with (1) shows that the rates of change of the charge q and the flux ϕ at time t are determined by $q(t)$, $\phi(t)$ and $u(t)$, which are the values of

the charge, flux and input *at time t*. Thus, if the initial charge $q(t_o)$ and the initial flux $\phi(t_o)$ are known together with the input $u(t)$, $t \geq t_o$, from time t_o on, the further evolution of the charge and flux is fully determined. This is more fully discussed in Section 5.4.

It is easy to see that if the charge, flux and input are given at any time t, also the output at that time is known. If the output y is the current i through the network, we have

$$y(t) = i(t) = \frac{1}{L}\phi(t).$$

If the output is the voltage v_L across the inductor it follows that

$$y(t) = v_L(t) = u(t) - \frac{R}{L}\phi(t) - \frac{1}{C}q(t).$$

Thus, if the initial charge and flux are known together with the subsequent behavior of the input, we do not only know the further behavior of the charge and the flux, but also that of the output.

Note that in this case the state is not a single, scalar quantity, but a vector-valued quantity with two components. Indeed, we may rewrite the differential equations (1) and (2) in matrix notation as

$$\frac{d}{dt}\begin{bmatrix} q(t) \\ \phi(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} q(t) \\ \phi(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{3}$$

Here

$$\frac{d}{dt}\begin{bmatrix} q(t) \\ \phi(t) \end{bmatrix} := \begin{bmatrix} \frac{dq(t)}{dt} \\ \frac{d\phi(t)}{dt} \end{bmatrix}.$$

The equation (3) is known as the *state differential equation* of the system. If the output of the network is the current,

$$y(t) = \begin{bmatrix} 0 & \frac{1}{L} \end{bmatrix} \begin{bmatrix} q(t) \\ \phi(t) \end{bmatrix}, \tag{4}$$

while if the output is the voltage across the inductor,

$$y(t) = \begin{bmatrix} -\frac{1}{C} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} q(t) \\ \phi(t) \end{bmatrix} + u(t). \tag{5}$$

The equation (4) or (5), respectively, is known as the *output equation* of the system.

Defining the *state vector*

$$x(t) := \begin{bmatrix} q(t) \\ \phi(t) \end{bmatrix}$$

the state differential equation and output equation of the system may be rewritten in the form

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{6a}$$

$$y(t) = Dx(t) + Eu(t), \tag{6b}$$

where the overdot denotes differentiation, and the matrices A and B are given by

$$A = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{R}{L} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For the output equation (4) the matrices D and E in (6b) are

$$D = \begin{bmatrix} 0 & \frac{1}{L} \end{bmatrix}, \qquad E = 0,$$

while for the output equation (5)

$$D = \begin{bmatrix} -\frac{1}{C} & -\frac{R}{L} \end{bmatrix}, \qquad E = 1.$$

Later in this chapter systems described by state differential and output equations of the form (6) are extensively analyzed.

We conclude by pointing out that the choice of the state often is not unique. For the RCL network, for instance, rather than the charge of the capacitor we may choose its voltage v_C, and instead of the flux contained by the inductor we may use the current i through it.

Exercise: Rederive the state differential and output equations of the RCL network when the state consists of v_C and i. ■

The state of *mechanical* systems is described by the *positions* and *velocities* of all the masses: If these are given, then the further motion of the system may be determined without requiring information on the past behavior of the system. Again we illustrate this with a specific system.

5.2.2 Example: Moon rocket. A rocket descends vertically to the surface of the moon, as sketched in Fig. 5.2. The elevation of the center of gravity of the rocket above the surface at time t (measured in upward direction) is $h(t)$, while the speed of ascent is $v(t)$. The thrust force of the rocket engine is in upward direction and given by $cu(t)$, where $u(t)$ is the amount of mass expelled per unit time and c the (constant) expulsion speed.

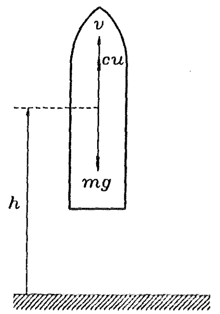


Figure 5.2. A rocket descends on the moon.

Given the time behavior of the thrust force from a fixed time t on, the descent of the rocket is determined by its initial position and velocity. These are the components of the state of the system. It is easy to find their rates of changes. For the elevation we have

$$\dot{h}(t) = v(t).$$

By Newton's law it follows that $m\dot{v}(t) = cu(t) - mg$, with m the mass of the rocket and g the acceleration of gravity on the moon, so that

$$\dot{v}(t) = \frac{c}{m}u(t) - g.$$

The English mathematician and natural philosopher Sir Isaac Newton (1642–1727) formulated the laws of gravity and motion and the elements of differential calculus.

Together these two equations form the state differential equation. Denoting $x_1(t) = h(t)$ and $x_2(t) = v(t)$ we have

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \dfrac{c}{m}\, u(t) - g. \end{cases}$$

These equations hold under the assumption that the mass of the rocket is *constant*. Actually, it is not, because of the expulsion of mass. A more accurate model therefore also includes the mass $m(t)$ as a component of the state. Because of the variable mass it is appropriate to replace the velocity $v(t)$ by the momentum $p(t) = m(t)v(t)$ as state variable. The rates of change of the state variables may now be expressed as

$$\dot{h}(t) = v(t) = \frac{p(t)}{m(t)},$$

$$\dot{p}(t) = cu(t) - m(t)g,$$

$$\dot{m}(t) = -u(t).$$

Defining $x_1(t) = h(t)$, $x_2(t) = p(t)$ and $x_3(t) = m(t)$, the state differential equation thus takes the form

$$\begin{cases} \dot{x}_1(t) = \dfrac{x_2(t)}{x_3(t)}, \\ \dot{x}_2(t) = cu(t) - gx_3(t), \\ \dot{x}_3(t) = -u(t). \end{cases}$$

If we consider the elevation of the rocket as its output, the output equation of the system is $y(t) = h(t)$, or

$$y(t) = x_1(t),$$

both when the mass is taken constant and time-varying. ∎

As a final example to illustrate the notion of state we consider a *digital network* consisting of delays, adders, and gains. If at any fixed time the contents of all the *delay elements* are known, it is not necessary to know anything about the past behavior. The contents of the delays thus constitute the *state* of the system. We consider a specific example.

5.2.3. Example: A tapped delay line. A tapped delay line, also called a *transversal filter*, is the digital network schematically represented in Fig. 5.3. The discrete-time input u is fed into a series connection of M unit delays. After multipli-

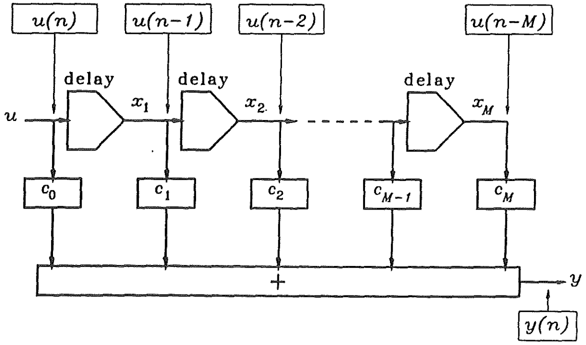


Figure 5.3. A tapped delay line. The boxed quantities are the signal values at time n.

cation by scalar coefficients the outputs of the delays are added to form the output y of the circuit.

To find the state of the system at time n we need know which quantities determine the current and future output values $y(n)$, $y(n+1)$, $\cdots$, if the current and future values of the input $u(n)$, $u(n+1)$, $\cdots$, are given. From the block diagram the required quantities may be seen to be the outputs $u(n-1)$, $u(n-2)$, $\cdots$, $u(n-M)$ of the delays. Thus, the M components of the state $x(n)$ at time n are

$$x_1(n) = u(n-1),$$

$$x_2(n) = u(n-2),$$

$$\cdots \quad \cdots \quad \cdots$$

$$x_M(n) = u(n-M).$$

The evolution of the state is determined by a set of first-order difference equations that is easily found as follows:

$$x_1(n+1) = u(n),$$

$$x_2(n+1) = u(n-1) = x_1(n),$$

$$x_3(n+1) = u(n-2) = x_2(n),$$

$$\cdots \quad \cdots \quad \cdots$$

$$x_M(n+1) = u(n-M+1) = x_{M-1}(n).$$

Defining the state vector

$$x(n) := \begin{bmatrix} x_1(n) \\ x_2(n) \\ \cdots \\ x_M(n) \end{bmatrix},$$

we may rewrite these equations in the form of the *state difference equation*

$$x(n + 1) = Ax(n) + Bu(n), \tag{7}$$

where the matrices A and B are given by

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}.$$

Given the state $x(n)$ and the input $u(n)$ at time n the output $y(n)$ at that same time follows as

$$\begin{aligned} y(n) &= c_0u(n) + c_1u(n - 1) + \cdots + c_Mu(n - M) \\ &= c_1x_1(n) + c_2x_2(n) + \cdots + c_Mx_M(n) + c_0u(n). \end{aligned}$$

This *output equation* may be rewritten as

$$y(n) = Cx(n) + Du(n), \tag{8}$$

where

$$C = [c_1 \quad c_2 \quad \cdots \quad c_M], \qquad D = c_0.$$

In the sequel we pay considerable attention to state systems described by a state difference equation of the form (7) and output equation (8). ∎

The procedure of the examples may be summarized as follows:

1. First determine those variables that summarize the past, referred to as the *state variables*.
2. Next find first-order difference or differential equations that describe the evolution of the state.
3. Finally express the output at any given time in the state and input at that same time.

State Systems

After having introduced the notion of state intuitively and by examples we turn to a formal definition of *input-output-state* (IOS) systems. The idea is to introduce, in addition to the input and output, a third signal—the state—that has the property that if

the state at a fixed time t is given, no values of the input prior to t are needed to d termine the output and state after time t.

5.2.4. Definition: Input-output-state system. Let $\mathbb{T}$ be a time axis, and $\mathcal{U}$ $\mathcal{Y}$ and $\mathcal{X}$ sets of time signals defined on the time axis $\mathbb{T}$. Elements of $\mathcal{U}$, $\mathcal{Y}$ and are called *input signals, output signals* and *state signals*, respectively. Then an *inpu output-state system* is defined by a subset $\mathcal{R}$ of $\mathcal{U} \times \mathcal{Y} \times \mathcal{X}$, called the *rule* or *rela tion* of the system, that has the following property.

Suppose that (u', y', x') and (u'', y'', x'') both belong to $\mathcal{R}$ such that

$$x'(t_o) = x''(t_o) \tag{9}$$

for some $t_o \in \mathbb{T}$. Then the following hold:

(a) *State matching property:* Define the *concatenated* signals u, y and x by

$$u(\tau) = \begin{cases} u'(\tau) & \text{for } \tau < t_o, \\ u''(\tau) & \text{for } \tau \geq t_o, \end{cases} \qquad y(\tau) = \begin{cases} y'(\tau) & \text{for } \tau < t_o, \\ y''(\tau) & \text{for } \tau \geq t_o, \end{cases}$$

$$x(\tau) = \begin{cases} x'(\tau) & \text{for } \tau < t_o, \\ x''(\tau) & \text{for } \tau \geq t_o, \end{cases}$$

for $\tau \in \mathbb{T}$. Then if (9) holds (u, y, x) forms an input-output-state triple, that is, $(u, y, x) \in \mathcal{R}$.

(b) *Causality:* If $u'(\tau) = u''(\tau)$ for $t_o \leq \tau < t$ and $\tau \in \mathbb{T}$, then if (9) holds

$$x'(\tau) = x''(\tau) \quad \text{for } t_o \leq \tau \leq t,$$

$$y'(\tau) = y''(\tau) \quad \text{for } t_o \leq \tau < t,$$

for $\tau \in \mathbb{T}$. ∎

In the sequel we often simply write *state system* for input-output-state system.

The matching property implies that if two different "pasts" of input, output and state lead to the same state at some given time, both "futures" constitute a valid continuation. Causality means that if the initial state is the same and the inputs coincide during a certain initial time interval, the outputs and states also coincide on this interval.

The signal range X of the state signal is called the *state space* of the system.

5.2.5 Example: State Systems.

(a) *RCL network.* The RCL network of Example 5.2.1 is a state system whose state has the components $x_1 = q$ and $x_2 = \phi$. The state space of the system is $X = \mathbb{R}^2$.

(b) *Moon rocket*. The moon rocket of Example 5.2.2 is a state system. Accounting for the time-dependent behavior of the mass of the rocket the state of the system has the three components $x_1 = h$, $x_2 = p$ and $x_3 = m$. The state space of the system is $X = \mathbb{R}^3$.

(c) *Tapped delay line*. Also the tapped delay line of Example 5.2.3 is a state system. Its state x has M components, so that the state space of the system is $X = \mathbb{R}^M$.

(d) *Binary tapped delay line*. Suppose that in the tapped delay line of 5.2.3 all variables, including the coefficients, are *binary*, i.e., take their values in $\mathbb{B} = \{0, 1\}$, and that the additions and multiplications are defined modulo 2. Then the state x still has the M components as in Fig. 5.3, but the state space now is $\mathbb{B}^M$. ∎

The examples we saw so far all had a state vector with a finite number of components (i.e., their state space was of the form $X = \mathbb{R}^N$). The state space of the following example consists of *functions*.

5.2.6. Example: Continuous-time delay. Consider the continuous-time system whose IO relation is given by

$$y(t) = u(t - \theta), \qquad t \in \mathbb{R},$$

with θ a positive number. Given the input $u(\tau)$ for $\tau \geq t$, we also need know the preceding input segment $u(\tau)$, $t - \theta \leq \tau < t$, to determine the output y from time t on. Hence, the state at time t is the function segment $x(t)$ defined by

$$(x(t))(\theta) = u(t - \theta + \tau), \qquad 0 \leq \tau < \theta,$$

as illustrated in Fig. 5.4. The state space X of this system consists of all functions defined on the interval $[0, \theta)$. ∎

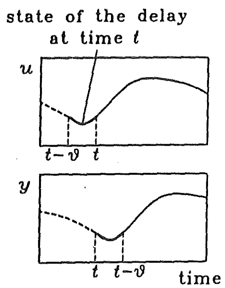


Figure 5.4. State of the continuous-time delay.

State Difference and Differential Systems

In the sequel we restrict ourselves to two special types of state systems, one discrete-time, the other continuous-time. They are the most common although by no means the only important types of state systems, and are characterized by the fact that their state space X is *finite-dimensional*, with $X = \mathbb{R}^N$ or $X = \mathbb{C}^N$. The RCL network, the moon rocket and the digital network of Examples 5.2.1, 5.2.2, and 5.2.3 all belong to this class of state systems.

First consider a state system as in 5.2.4 defined on the discrete time axis $\mathbb{T} = \mathbb{Z}$. Then if for some $n \in \mathbb{Z}$ the state $x(n)$ and input $u(n)$, both at time n, are given, by property 5.2.4(b) the state $x(n + 1)$ at time $n + 1$ and the output $y(n)$ at time n are fully determined. Hence, there exist functions f and g such that

$$\begin{aligned} x(n + 1) &= f(n, x(n), u(n)), \\ y(n) &= g(n, x(n), u(n)), \qquad n \in \mathbb{Z}. \end{aligned}$$

The first of these equations is the *state difference equation* of the system and the second the *output equation*. If the state difference equation is known, the evolution of the state from any initial state is fully determined by successive application of the difference equation. Once the behavior of the state has been found, that of the output follows by the output equation.

The state difference equation describes the evolution of the state over the shortest possible time span on the discrete time axis $\mathbb{Z}$. The state difference and output equations often may directly be obtained from the physical or other fundamental laws that govern the system. Systems of this type where at any given time the state belongs to the finite-dimensional state space $X = \mathbb{R}^N$ or $X = \mathbb{C}^N$, and the input and output signal ranges are also finite-dimensional spaces, are called *state difference systems*. The state difference and output equations for such systems may more explicitly be rendered as the *set* of equations

$$\begin{cases} x_1(n + 1) = f_1(n, x_1(n), x_2(n), \cdots, x_N(n), u_1(n), u_2(n), \cdots, u_K(n)), \\ x_2(n + 1) = f_2(n, x_1(n), x_2(n), \cdots, x_N(n), u_1(n), u_2(n), \cdots, u_K(n)), \\ \cdots \quad \cdots \quad \cdots \\ x_N(n + 1) = f_N(n, x_1(n), x_2(n), \cdots, x_N(n), u_1(n), u_2(n), \cdots, u_K(n)), \end{cases}$$

$$\begin{cases} y_1(n) = g_1(n, x_1(n), x_2(n), \cdots, x_N(n), u_1(n), u_2(n), \cdots, u_K(n)), \\ y_2(n) = g_2(n, x_1(n), x_2(n), \cdots, x_N(n), u_1(n), u_2(n), \cdots, u_K(n)), \\ \cdots \quad \cdots \quad \cdots \\ y_M(n) = g_M(n, x_1(n), x_2(n), \cdots, x_N(n), u_1(n), u_2(n), \cdots, u_K(n)), \end{cases}$$

for $n \in \mathbb{Z}$. Here the x_i are the components of x, the u_i those of u and the y_i those of y. Also, the f_i are the components of the vector-valued function f, and the g_i those of g. The tapped delay line of Example 5.2.3 is an example of a state difference system.

The continuous-time equivalent of a state difference system is a *state differential system*. This is a state system defined on the continuous time axis $\mathbb{T} = \mathbb{R}$ with state space $X = \mathbb{R}^N$ or $X = \mathbb{C}^N$ and finite-dimensional input and output signal ranges, and is described by the *state differential* and *output* equations

$$\dot{x}(t) = f(t, x(t), u(t)),$$
$$y(t) = g(t, x(t), u(t)), \qquad t \in \mathbb{R}.$$

As before, the overdot denotes differentiation. The state differential equation again describes the evolution of the state on the shortest possible time span. Under suitable conditions on the (vector-valued) function f, which are briefly touched upon in Section 5.4, the state differential equation uniquely determines the behavior of the state starting from any initial state if the input is known. The state differential equation and continuous-time output equation may be expressed in component form as

$$\begin{cases} \dot{x}_1(t) = f_1(t, x_1(t), x_2(t), \cdots, x_N(t), u_1(t), u_2(t), \cdots, u_K(t)), \\ \dot{x}_2(t) = f_2(t, x_1(t), x_2(t), \cdots, x_N(t), u_1(t), u_2(t), \cdots, u_K(t)), \\ \cdots \quad \cdots \quad \cdots \\ \dot{x}_N(t) = f_N(t, x_1(t), x_2(t), \cdots, x_N(t), u_1(t), u_2(t), \cdots, u_K(t)), \end{cases}$$

$$\begin{cases} y_1(t) = g_1(t, x_1(t), x_2(t), \cdots, x_N(t), u_1(t), u_2(t), \cdots, u_K(t)), \\ y_2(t) = g_2(t, x_1(t), x_2(t), \cdots, x_N(t), u_1(t), u_2(t), \cdots, u_K(t)), \\ \cdots \quad \cdots \quad \cdots \\ y_M(t) = g_M(t, x_1(t), x_2(t), \cdots, x_N(t), u_1(t), u_2(t), \cdots, u_K(t)). \end{cases}$$

Again, the x_i are the components of x, the u_i those of u, and the y_i those of y, while the f_i are the components of the vector-valued function f and the g_i those of g. The RCL network of Example 5.2.1 and the moon rocket of Example 5.2.2 are state differential systems. The continuous-time delay of Example 5.2.6 is *not* a state differential system because its state space is not finite-dimensional.

State Transition Map

Consider a general input-output-state system as defined in 5.2.4. Part (b) of the definition implies that if the state $x(t_o)$ of the system at some initial time t_o and the input u are given, the state $x(t)$ at time t with $t > t_o$ is uniquely determined. Hence, there exists a map, denoted s, such that

$$x(t) = s(t, t_o, x(t_o), u).$$

Since the map s describes how the input u takes the system from the state $x(t_o)$ at time t_o to the state $x(t)$ at time t, it is called the *state transition map* of the system.

The state transition map has the following properties, which may easily verified:

(a) *Consistency:* For any $t \in \mathbb{T}$, any $x \in X$ and any $u \in \mathcal{U}$

$$s(t, t, x, u) = x.$$

Consistency simply means that if no time elapses, the system remains in tl same state.

(b) *Semigroup property:* Let t_0, t_1 and t_2 be three time instants belonging to the tin axis $\mathbb{T}$ such that $t_0 \leq t_1 \leq t_2$. Then for all $x_o \in X$ and all $u \in \mathcal{U}$

$$s(t_2, t_0, x_o, u) = s(t_2, t_1, s(t_1, t_0, x_o, u), u).$$

This property implies that taking the system from some initial state at time t_0 t some intermediate state at time t_1 and from this to a final state at time t_2 result in the same final state as taking the system directly from the initial state at tim t_0 to the final state at time t_2.

(c) *Non-anticipativity:* Let t_o and t be two time instants such that $t_o < t$, and u' and u'' two inputs that coincide on the interval $[t_o, t)$, i.e., $u'(\tau) = u''(\tau)$ for $t_o \leq \tau < t$ and $\tau \in \mathbb{T}$. Then for any $x_o \in X$

$$s(t, t_o, x_o, u') = s(t, t_o, x_o, u'').$$

Non-anticipativity means that the transition of the state from time t_o to time t only depends on the input during the intervening interval, and *not* on the input at any other time.

We illustrate the state transition map for the RC network.

5.2.7. Example: State transition map of the RC network. The circuit diagram of the RC network is repeated in Fig. 5.5. The state of the system consists of the charge q of the capacitor. By Kirchhoff's current law $\dot{q}(t) = i(t)$, with i the current through the network. Since by Kirchhoff's voltage law $u(t) = v_R(t) + v_C(t) = Ri(t) + q(t)/C$ it follows that $i(t) = u(t)/R - q(t)/RC$, so that

$$\dot{q}(t) = -\frac{1}{RC}q(t) + \frac{1}{R}u(t), \qquad t \in \mathbb{R}.$$

Replacing q with x results in the state differential equation

$$\dot{x}(t) = -\frac{1}{RC}x(t) + \frac{1}{R}u(t), \qquad t \in \mathbb{R}. \tag{10}$$

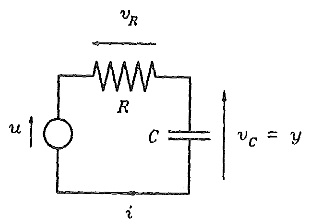


Figure 5.5. RC network.

The corresponding output equation is

$$y(t) = \frac{q(t)}{C} = \frac{1}{C}x(t).$$

For a given initial condition $x(t_o)$ and given input u the solution of the scalar first-order state differential equation (10) may be verified to be given by

$$x(t) = e^{-\frac{t-t_o}{RC}}x(t_o) + \frac{1}{R}\int_{t_o}^{t} e^{-\frac{t-\tau}{RC}}u(\tau)\,d\tau, \qquad t \geq t_o.$$

This expression explicitly represents the state transition map. The consistency, semi-group and non-anticipativity properties are easily seen to hold. ∎

Linearity of State Systems

We conclude this section with brief discussions of the *linearity* and *time-invariance* of input-output-state systems. Linearity, as in the case of input-output systems, is equivalent to linearity of the relation $\mathfrak{R}$ in 5.2.4 that defines the system.

5.2.8. Definition: Linearity of input-output-state systems. The input-output-state system with relation $\mathfrak{R} \subset \mathcal{U} \times \mathcal{Y} \times \mathcal{X}$ is *linear* if $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{X}$ are linear spaces and the relation $\mathfrak{R}$ is a linear subspace. ∎

Linearity of a state system means that if $(u', y', x') \in \mathfrak{R}$ and $(u'', y'', x'') \in \mathfrak{R}$ are two input-output-state triples, any linear combination of these is also an input-output-state triple.

It may easily be proved that state difference systems are linear if their state difference and output equations are both linear, and that state differential systems are linear if their state differential and output equations are both linear:

5.2.9. Summary: Linearity of state difference systems and state differential systems.

The state difference system with state difference and output equations

$$x(n+1) = A(n)x(n) + B(n)u(n),$$
$$y(n) = C(n)x(n) + D(n)u(n),$$

$n \in \mathbb{Z}$, with $A(n)$, $B(n)$, $C(n)$, $D(n)$, $n \in \mathbb{Z}$, matrices of appropriate dimensions whose elements are real- or complex-valued functions of time, is linear.

The state differential system with state differential and output equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t),$$

$t \in \mathbb{R}$, with $A(t)$, $B(t)$, $C(t)$, $D(t)$, $t \in \mathbb{R}$, matrices of appropriate dimensions whose elements are real- or complex-valued bounded and continuous functions of time, is linear. ∎

As seen in Section 5.4, the boundedness and continuity of the matrix functions A, B, C and D guarantee the existence of solutions to the state differential equation.

5.2.10. Examples: Linear state difference and differential systems. It is easily recognized that the state difference and output equations of the tapped delay line of Example 5.2.3 are of the type $x(n+1) = Ax(n) + Bu(n)$, $y(n) = Cx(n) + Du(n)$, with A, B, C and D constant matrices. Hence, the tapped delay line is linear.

The RCL network of Example 5.2.1 and the RC network of Example 5.2.7 have a state differential equation and output equation of the form $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$, with A, B, C and D constant matrices. Hence, also these systems are linear.

The state differential and output equations of the moon rocket of Example 5.2.2 (accounting for the variable mass) are *not* linear. Indeed, it may easily be shown that the system is nonlinear. ∎

Time-Invariance of State Systems

It remains to discuss time-invariance of state systems. As in the case of input-output systems, time-invariance amounts to shift-invariance of the relation that defines the system.

5.2.11. Definition: Time-invariance of input-output-state systems. The input-output-state system with relation $\mathfrak{R} \subset \mathcal{U} \times \mathcal{Y} \times \mathcal{X}$ defined on an infinite or right semi-infinite time axis is time-invariant if $\mathfrak{R}$ is shift-invariant, that is, if $(u, y, x) \in \mathfrak{R}$ then $(\sigma^\theta u, \sigma^\theta y, \sigma^\theta x) \in \mathfrak{R}$ for any allowable time shift θ. ∎

Time-invariance of the state system implies that if any input-output-state triple is shifted over any amount of time, it still is an input-output-state triple. Time-invariance means that the dynamic properties of the system do not change with time.

State difference and differential systems are time-invariant if the right-hand sides of the state difference or differential equation, respectively, and that of the output equation depend on time only through the state x and the input u and not directly:

5.2.12. Summary: Time-invariance of state difference and differential systems.

The discrete-time system described by the state difference and output equations

$$x(n+1) = f(x(n), u(n)),$$
$$y(n) = g(x(n), u(n)), \qquad n \in \mathbb{Z},$$

is time-invariant.

The continuous-time system described by the state differential and output equations

$$\dot{x}(t) = f(x(t), u(t)),$$
$$y(t) = g(x(t), u(t)), \qquad t \in \mathbb{R},$$

is time-invariant. ∎

Combining 5.2.11 and 5.2.12 we obtain the following result.

5.2.13. Summary: Linear time-invariant state difference and differential systems.

The state difference system with state difference and output equations

$$x(n+1) = Ax(n) + Bu(n),$$
$$y(n) = Cx(n) + Du(n), \qquad n \in \mathbb{Z},$$

with A, B, C and D constant matrices, is both linear and time-invariant.

The state differential system with state differential and output equations

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t), \qquad t \in \mathbb{R},$$

with A, B, C and D constant matrices, is both linear and time-invariant. ∎

5.2.14. Example: Time-invariant state difference and differential systems. The tapped delay line of Example 5.2.3 has a state difference and output equation of the form $x(n+1) = Ax(n) + Bu(n)$, $y(n) = Cx(n) + Du(n)$, $n \in \mathbb{Z}$, with A, B, C and D constant matrices. Hence, this system is both time-invariant and linear.

The RCL network of Example 5.2.1 and the RC network of Example 5.2.7 have state differential and output equations of the form $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$, $t \in \mathbb{R}$, with A, B, C and D constant matrices, and hence are also both linear and time-invariant.

The moon rocket of Example 5.2.2 is time-invariant but not linear. If the constant c would change with time the system would not only be nonlinear but also time-varying. ∎

5.2.15. Review: Sampled state difference systems. Sampled state difference systems are input-output-state systems defined on the time axis $\mathbb{Z}(T)$. They are described by a state difference and output equation of the form

$$x(t+T) = f(t, x(t), u(t)),$$
$$y(t) = g(t, x(t), u(t)), \qquad t \in \mathbb{Z}(T).$$

Sampled state difference systems are linear if their state difference and output equations are of the form

$$x(t+T) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t), \qquad t \in \mathbb{Z}(T),$$

with A, B, C, and D possibly time-dependent matrices of suitable dimensions. Sampled state difference systems described by

$$x(t+T) = f(x(t), u(t)),$$
$$y(t) = g(x(t), u(t)), \qquad t \in \mathbb{Z}(T),$$

are time-invariant. Linear time-invariant sampled state difference systems, finally, have a state difference and output equation of the form

$$x(t+T) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t), \qquad t \in \mathbb{Z}(T),$$

with A, B, C, and D *constant* matrices. ∎

5.3 REALIZATION OF DIFFERENCE AND DIFFERENTIAL SYSTEMS AS STATE SYSTEMS

In this section we study how linear difference and differential systems, which are described by a difference or differential equation that relates input and output, may be represented as state systems. Because as we shall shortly see state difference and differential systems may easily be implemented in hardware or software, these results are important when it comes to actually building difference and differential systems. Moreover, constructing a state representation helps to study and understand its properties.

Implementation of State Difference and Differential Systems

We first discuss the implementation of *state difference systems*. For this we assume the availability of three basic building blocks, namely, *unit time delays, function generators,* and *time clocks*. These building blocks may exist as *hardware* (i.e., in the form of physical devices), or as *software* (i.e., computer routines that emulate the physical function).

A *unit time delay* may be represented as in Fig. 5.6(a). It accepts a scalar or vector-valued discrete-time signal u as input and delays it by one time unit, so that its output is

$$y(n) = u(n-1), \qquad n \geq n_o, \qquad n \in \mathbb{Z}.$$

At start-up time n_o the unit delay requires an initial value $y(n_o) = y_o$, which is also indicated in Fig. 5.6(a).

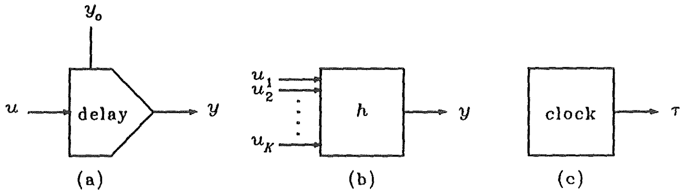


Figure 5.6. Building blocks for discrete-time state systems. Left: unit delay. Middle: function generator. Right: time clock.

A *function generator* may be depicted as in Fig. 5.6(b). It accepts a number of scalar or vector-valued discrete-time signals $u_1, u_2, \cdots, u_K$ as inputs, and produces the output y given by

$$y(n) = h(u_1(n), u_2(n), \cdots, u_K(n)), \qquad n \in \mathbb{Z},$$

with h a given function.

A *time clock,* finally, as shown in Fig. 5.6(c), is a system without input whose output is a discrete-time signal τ such that

$$\tau(n) = n, \qquad n \in \mathbb{Z}.$$

Given these building blocks, a state difference system with state difference and output equations

$$\begin{aligned} x(n+1) &= f(n, x(n), u(n)), \\ y(n) &= g(n, x(n), u(n)), \qquad n = n_o, n_o + 1, \cdots, \end{aligned}$$

may be realized as in the block diagram of Fig. 5.7. At any instant of time n one may measure at each point in the block diagram the value of the corresponding signal at that time. From the block diagram we see that at time n

$$(\sigma x)(n) = f(\tau(n), x(n), u(n)),$$

or

$$x(n+1) = f(n, x(n), u(n)).$$

Moreover,

$$y(n) = g(\tau(n), x(n), u(n)) = g(n, x(n), u(n)).$$

In practice, since the input to the unit delay is a vector-valued signal x with components $x_1, x_2, \cdots, x_N$, the unit delay consists of a *parallel bank of delays*, each of which accepts a scalar signal as input. Similarly, the function generators f and possibly also g are parallel banks of function generators, each accepting several signals simultaneously as input and producing one of the components of x or y, respectively, as output.

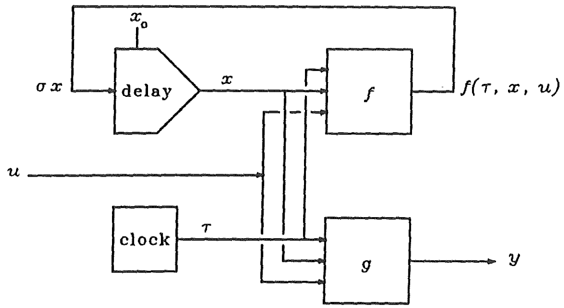


Figure 5.7. Implementation of a state difference system.

Sampled state difference equations as discussed in 5.2.15 are defined on the time axis $\mathbb{T} = \mathbb{Z}(T)$, and may be realized as in Fig. 5.7 with delays that retard by one sampling interval T.

For the realization of state differential systems the role of the unit delay is taken by an *integrator*. An integrator accepts a scalar or vector-valued signal u as input and produces the output

$$y(t) = y(t_o) + \int_{t_o}^{t} u(\theta)\, d\theta, \qquad t \geq t_o.$$

The initial value $y(t_o)$ need be set at the initial time t_o. Differentiating, we find that input and output are related as

$$\dot{y}(t) = u(t), \qquad t \geq t_o.$$

Integrators are represented in a block diagram as in Fig. 5.8.

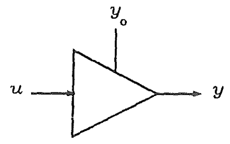


Figure 5.8. An integrator.

The state differential system with state differential and output equations

$$\dot{x}(t) = f(t, x(t), u(t)),$$
$$y(t) = g(t, x(t), u(t)), \qquad t \geq t_o, \qquad t \in \mathbb{R},$$

may be implemented using an integrator, function generators and a time clock as in Fig. 5.9. The time clock now runs in continuous time. Also here, an integrator that operates on vector-valued signals in practice is constructed as a parallel bank of scalar integrators.

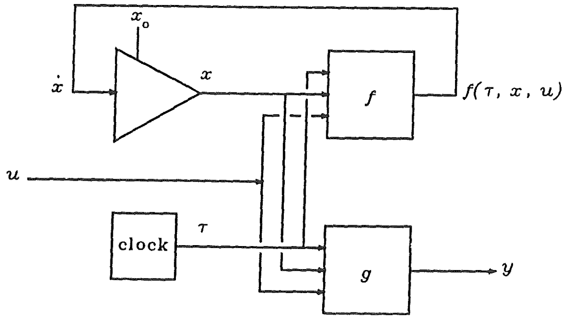


Figure 5.9. Implementation of a state differential system.

5.3.1. Example: Simulation of the moon rocket. Suppose that we wish to build an implementation of the equations that describe the moon rocket of Example 5.2.2. Such an implementation is called a *simulation*, and allows studying the dynamic behavior of the rocket without constructing the rocket itself. The state differential equation is given by

$$\dot{x}_1(t) = \frac{x_2(t)}{x_3(t)},$$
$$\dot{x}_2(t) = cu(t) - gx_3(t),$$
$$\dot{x}_3(t) = -u(t),$$

while the output equation is

$$y(t) = x_1(t).$$

Figure 5.10 shows a block diagram of the implementation. It uses three scalar integrators and three function generators. Two of these function generators are "gains," which simply multiply their scalar input by a fixed constant. The third is a "divider," whose output is the quotient x_2/x_3 of its scalar inputs x_2 and x_3. The block diagram also includes an adder, which is another special type of function generator. Because the implementation has three integrators, three initial values are needed at start-up time. ∎

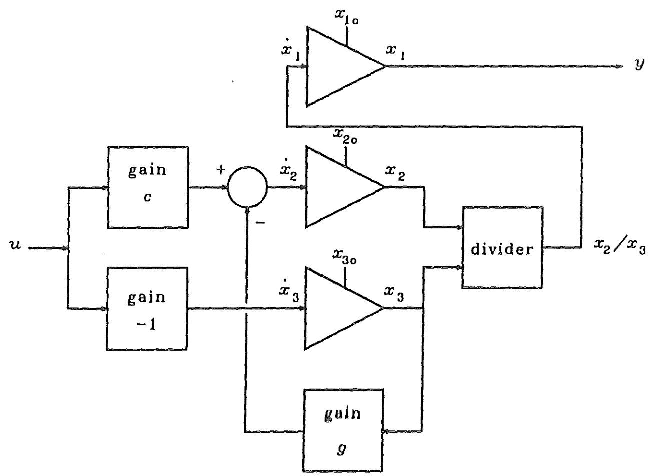


Figure 5.10. Implementation of the state equations of the moon rocket.

State Realization of Linear Difference and Differential Systems: Examples

To implement a difference or differential system we need a state realization of the system. A state realization of a linear constant coefficient difference or differential system may be found by the *matching property* of the state as defined in 5.2.4. We illustrate this by means of two examples.

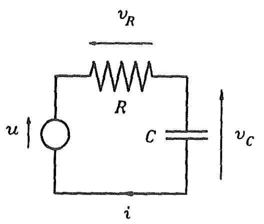


Figure 5.12. RC network.

Suppose now that the output y is not the voltage v_C across the capacitor but the voltage v_R across the resistor. Since $v_C = u - v_R$, the differential equation for this output follows by replacing y in (4) by $u - y$, resulting in the differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{du(t)}{dt}, \qquad t \in \mathbb{R}. \tag{5}$$

Again we use the state matching property to identify the state of this system. Suppose that (u', y') and (u'', y'') are two IO pairs satisfying the differential equation (5). Denoting the corresponding state signals as x' and x'', the concatenated pair (u, y) defined by

$$u(t) = \begin{cases} u'(t) & \text{for } t < t_o, \\ u''(t) & \text{for } t \geq t_o, \end{cases} \qquad t \in \mathbb{R},$$

$$y(t) = \begin{cases} y'(t) & \text{for } t < t_o, \\ y''(t) & \text{for } t \geq t_o, \end{cases} \qquad t \in \mathbb{R},$$

is an IO pair if $x'(t_o) = x''(t_o)$. Rewriting y as

$$y(t) = [1 - \mathbb{1}(t - t_o)]y'(t) + \mathbb{1}(t - t_o)y''(t), \qquad t \in \mathbb{R},$$

it follows after differentiation that

$$\frac{dy(t)}{dt} = [1 - \mathbb{1}(t - t_o)]\frac{dy'(t)}{dt} + \mathbb{1}(t - t_o)\frac{dy''(t)}{dt} + [y''(t_o) - y'(t_o)]\delta(t - t_o), \qquad t \in \mathbb{R}.$$

Similarly we have

$$\frac{du(t)}{dt} = [1 - \mathbb{1}(t - t_o)]\frac{du'(t)}{dt} + \mathbb{1}(t - t_o)\frac{du''(t)}{dt} + [u''(t_o) - u'(t_o)]\delta(t - t_o), \qquad t \in \mathbb{R}.$$

Substitution of y and u into the differential equation (5) and using the fact that both (u', y') and (u'', y'') satisfy the differential equation we find that (u, y) is an IO pair if

$$[y''(t_o) - y'(t_o)]\delta(t - t_o) = [u''(t_o) - u'(t_o)]\delta(t - t_o),$$

or

$$y''(t_o) - y'(t_o) = u''(t_o) - u'(t_o).$$

Rearrangement shows that (u, y) is an IO pair, that is, $x'(t_o) = x''(t_o)$, if

$$y'(t_o) - u'(t_o) = y''(t_o) - u''(t_o).$$

Hence, the state of the system may be chosen as

$$x(t) = y(t) - u(t), \qquad t \in \mathbb{R},$$

because then $x'(t_o) = x''(t_o)$ guarantees the concatenated pair (u, y) to be an IO pair. The state differential equation follows from (5) as

$$\dot{x}(t) = \frac{dy(t)}{dt} - \frac{du(t)}{dt} = -\frac{1}{RC}y(t) = -\frac{1}{RC}[x(t) + u(t)]$$
$$= -\frac{1}{RC}x(t) - \frac{1}{RC}u(t), \qquad t \in \mathbb{R},$$

while the output equation is

$$y(t) = x(t) + u(t), \qquad t \in \mathbb{R}.$$

An implementation of the system is given in Fig. 5.13.

We observe from the circuit diagram of Fig. 5.12 that $x = y - u = v_R - u = -v_C$. Hence, within a factor -1 the state equals the voltage across the capacitor, which in turn within a factor $1/C$ equals the charge q of the capacitor. This confirms the physical argument of 5.2.7 that the charge q may be chosen as the state of the network. ∎

State Realization of Linear Difference and Differential Systems

We now consider quite generally the state realization of difference systems described by the linear constant coefficient difference equation

5.3.2. Examples: State realization of difference and differential systems.

(a) *Exponential smoother*. The exponential smoother is described by the difference equation

$$y(n+1) = ay(n) + (1-a)u(n+1), \qquad n \in \mathbb{Z}. \tag{1}$$

To determine the state of the smoother we use the *state matching* property of 5.2.4. This property implies the following. Suppose that (u', y') and (u'', y'') are two input-output pairs of the smoother. Then for any $n_o \in \mathbb{Z}$ the concatenated pair (u, y) defined by

$$u(n) = \begin{cases} u'(n) & \text{for } n < n_o, \\ u''(n) & \text{for } n \geq n_o, \end{cases} \qquad n \in \mathbb{Z}, \tag{2a}$$

$$y(n) = \begin{cases} y'(n) & \text{for } n < n_o \\ y''(n) & \text{for } n \geq n_o, \end{cases} \qquad n \in \mathbb{Z}, \tag{2b}$$

is an IO pair provided $x'(n_o) = x''(n_o)$, where x' is the state signal corresponding to the pair (u', y') and x'' that corresponding to (u'', y''). Hence, we may identify the state by recognizing what constraint the requirement that (u, y) be an IO pair imposes on (u', y') and (u'', y'').

Substitution of (u, y) as given by (2) into the difference equation (1) requires distinguishing the cases $n < n_o - 1$, $n = n_o - 1$ and $n > n_o$. We successively obtain

$$\begin{aligned} n < n_o - 1: &\quad y'(n+1) = ay'(n) + (1-a)u'(n+1) \\ n = n_o - 1: &\quad y''(n_o) = ay'(n_o - 1) + (1-a)u''(n_o), \\ n > n_o - 1: &\quad y''(n+1) = ay''(n) + (1-a)u''(n+1). \end{aligned} \tag{3}$$

The equality for $n < n_o - 1$ is satisfied because by assumption (u', y') is an IO pair. Similarly, the equality for $n > n_o - 1$ is satisfied because by assumption also (u'', y'') is an IO pair. From (3) it follows that the two IO pairs need be related by

$$ay'(n_o - 1) = y''(n_o) - (1-a)u''(n_o).$$

Since (u'', y'') satisfies the difference equation this may be rewritten in the form

$$ay'(n_o - 1) = ay''(n_o - 1).$$

Now let

$$x'(n_o) = ay'(n_o - 1), \qquad x''(n_o) = ay''(n_o - 1).$$

Then the condition $x'(n_o) = x''(n_o)$ guarantees (u, y) to be an IO pair. It follows that the state of the exponential smoother (1) is given by

$$x(n) = ay(n-1), \qquad n \in \mathbb{Z}.$$

The resulting state difference and output equations for the exponential smoother may easily be established. We have from (1)

$$\begin{aligned} x(n+1) &= ay(n) = a^2y(n-1) + a(1-a)u(n) \\ &= ax(n) + a(1-a)u(n), \qquad n \in \mathbb{Z}, \end{aligned}$$

which constitutes the state difference equation. On the other hand, it follows from the input-output difference equation that

$$\begin{aligned} y(n) &= ay(n-1) + (1-a)u(n) \\ &= x(n) + (1-a)u(n), \qquad n \in \mathbb{Z}, \end{aligned}$$

which is the output equation of the state representation of the smoother. An implementation of the state representation is given in Fig. 5.11.

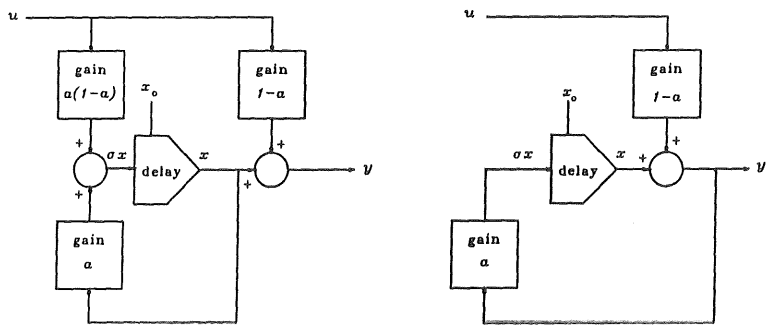


Figure 5.11. Implementation of the exponential smoother. Left: a direct implementation of the state difference and output equations. Right: a more efficient implementation using fewer coefficients.

(b) *RC network*. In Example 1.2.7 we have seen that if the output y of the RC network of Fig. 5.12 is the voltage across the capacitor, the network is described by the differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}u(t), \qquad t \in \mathbb{R}. \tag{4}$$

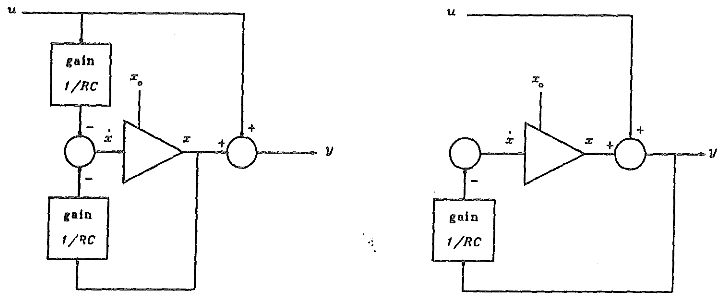


Figure 5.13. Implementation of the RC network with the output across the resistor as output. Left: a direct implementation of the state differential and output equations. Right: a more efficient implementation using fewer coefficients.

$$Q(\sigma)y = P(\sigma)u,$$

and differential systems described by

$$Q(D)y = P(D)u.$$

Q and P are the polynomials

$$Q(\lambda) = q_0 + q_1\lambda + \cdots + q_N\lambda^N,$$
$$P(\lambda) = p_0 + p_1\lambda + \cdots + p_M\lambda^M.$$

Without loss of generality we assume that $q_N = 1$. We furthermore suppose that $M \leq N$. By the latter assumption, without loss of generality we may take $M = N$, where in contrast to Chapter 4 the leading coefficient p_N may be zero.

The main result may be formulated as follows.

5.3.3. Summary: State realization of difference and differential systems.

The difference system described by the difference equation

$$Q(\sigma)y = P(\sigma)u,$$

where

$$Q(\lambda) = q_0 + q_1\lambda + \cdots + q_N\lambda^N,$$
$$P(\lambda) = p_0 + p_1\lambda + \cdots + p_N\lambda^N,$$

The differential system described by the differential equation

$$Q(D)y = P(D)u,$$

where

$$Q(\lambda) = q_0 + q_1\lambda + \cdots + q_N\lambda^N,$$
$$P(\lambda) = p_0 + p_1\lambda + \cdots + p_N\lambda^N,$$

with $q_N = 1$, has a state realization

$$x(n+1) = Ax(n) + Bu(n),$$
$$y(n) = Cx(n) + Du(n), \quad n \in \mathbb{Z},$$

such that

with $q_N = 1$, has a state realization

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t), \quad t \in \mathbb{R},$$

such that

$$A = \begin{bmatrix} -q_{N-1} & 1 & 0 & \cdots & 0 \\ -q_{N-2} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -q_1 & 0 & 0 & \cdots & 1 \\ -q_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} p_{N-1} - q_{N-1}p_N \\ p_{N-2} - q_{N-2}p_N \\ \cdots \\ p_1 - q_1p_N \\ p_0 - q_0p_N \end{bmatrix},$$

$$C = [1 \quad 0 \quad 0 \quad \cdots \quad 0], \qquad D = p_N. \qquad \blacksquare$$

In Figures 5.14 and 5.15 implementations are given of the state realizations using gains, adders, unit delays and integrators. The number of unit delays or integrators needed for the implementation equals the order N of the system.

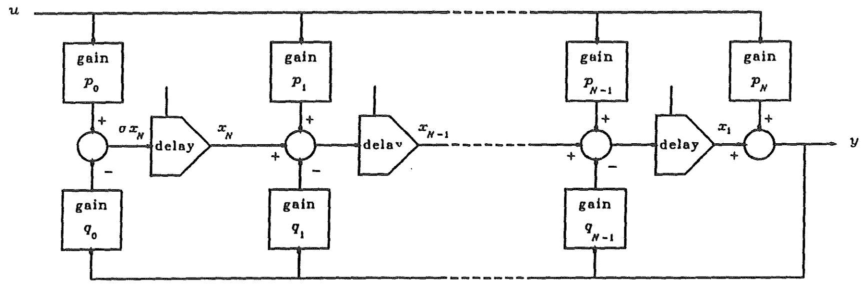


Figure 5.14. State realization of a difference system.

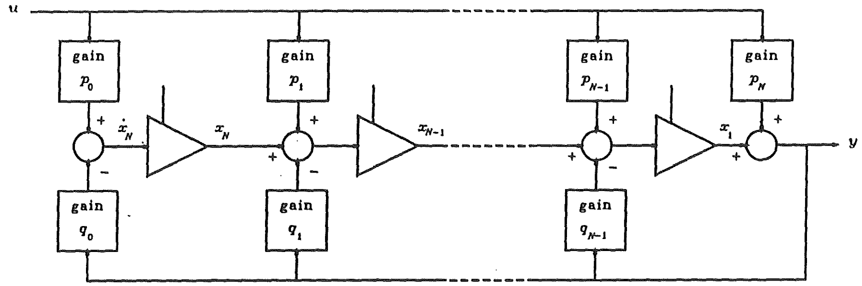


Figure 5.15. State realization of a differential system.

5.3.4. Proof of 5.3.3. We only give an outline of the proof. The details are straightforward but lengthy.

(a) *Discrete-time case*. To identify the state of the system described by the difference equation $Q(\sigma)y = P(\sigma)u$ we employ the state matching property of 5.2.4. N conditions are needed to make sure that the concatenation of two IO pairs at time n is again an IO pair. Inspection of the conditions shows that the corresponding N components of the state are

$$\begin{aligned}
x_1(n) &= p_0u(n-N) + p_1u(n-N+1) + \cdots + p_{N-1}u(n-1) \\
&\quad - q_0y(n-N) - q_1y(n-N+1) - \cdots - q_{N-1}y(n-1), \\
x_2(n) &= p_0u(n-N+1) + p_1u(n-N+2) + \cdots + p_{N-2}u(n-1) \\
&\quad - q_0y(n-N+1) - q_1y(n-N+2) - \cdots - q_{N-2}y(n-1), \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \\
x_N(n) &= p_0u(n-1) - q_0y(n-1).
\end{aligned}$$

From these definitions the state difference and output equations follow easily with the help of the input-output difference equation.

(b) *Continuous-time case*. Also in the continuous-time case the matching property is used. Concatenation of two IO pairs normally causes a step change, which after substitution into the input-output differential equation results in δ-functions and derivatives of δ-functions. Matching the coefficients of the δ-functions leads to N conditions, from which it follows that the components of the state at time t may be chosen as

$$\begin{aligned}
x_1(t) &= y(t) - p_Nu(t), \\
x_2(t) &= q_{N-1}y(t) + Dy(t) - p_{N-1}u(t) - p_NDu(t), \\
&\cdots \quad \cdots \quad \cdots \\
x_N(t) &= q_1y(t) + q_2Dy(t) + \cdots + D^{N-1}y(t) \\
&\quad - p_1u(t) - p_2Du(t) - \cdots - p_ND^{N-1}u(t).
\end{aligned}$$

The state differential and output equations follow easily from this. ∎

Given a difference system $Q(\sigma)y = P(\sigma)u$ or a differential system $Q(D)y = P(D)u$, the realizations of 5.3.3 are by no means the only state representations. In fact, as seen in Section 5.6, any nonsingular transformation of the state may serve as a state for the system.

We close this section with some further examples of state realizations.

5.3.5. Examples: State realizations of difference and differential systems.

(a) *Second-order smoother*. Consider the second-order smoother described by the difference equation

$$y(n+2) - a_1y(n+1) - a_0y(n) = b_2u(n+2) + b_1u(n+1), \qquad n \in \mathbb{Z},$$

introduced in Example 4.2.4(b). It follows that

$$Q(\lambda) = -a_0 - a_1\lambda + \lambda^2, \qquad P(\lambda) = b_1\lambda + b_2\lambda^2.$$

Since the order of the system is $N = 2$, the dimension of the state is also 2. Identifying $q_1 = -a_1$, $q_0 = -a_0$, $p_2 = b_2$, $p_1 = b_1$ and $p_0 = 0$, by 5.3.3 the system may be represented in state form as

$$\begin{aligned}
x(n+1) &= \begin{bmatrix} a_1 & 1 \\ a_0 & 0 \end{bmatrix} x(n) + \begin{bmatrix} b_1 + a_1b_2 \\ a_0b_2 \end{bmatrix} u(n), \\
y(n) &= [1 \quad 0]x(n) + b_2u(n), \qquad n \in \mathbb{Z}.
\end{aligned}$$

A block diagram of the implementation of the smoother is given in Fig. 5.16.

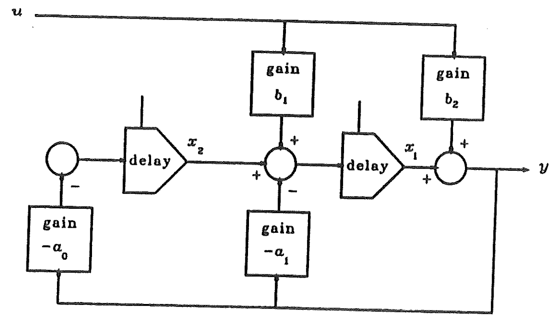


Figure 5.16. State space realization of the second-order smoother.

(b) *Approximate differentiator*. A differentiator is a continuous-time system with IO map $y = Du$. Its frequency response function is

$$\hat{h}(f) = j2\pi f, \qquad f \in \mathbb{R}.$$

As frequency increases, the magnitude of the frequency response function goes to infinity. This causes large amplification of any noise added to the input, which is always present and usually contains high frequencies. This difficulty may be avoided by using an *approximate* differentiator, whose frequency response function behaves like that of the differentiator at low frequencies, but stops increasing or even starts decreasing at high frequencies. An example of an approximate differentiator is a system with frequency response function

$$\hat{h}(f) = \frac{j2\pi f}{(1 + j2\pi f/\omega_0)^2}, \qquad f \in \mathbb{R},$$

with ω_0 a positive number. The system behaves like a differentiator more or less up to the frequency $\omega_0/2\pi$. The function $\hat{h}$ is the frequency response function

$$\hat{h}(f) = \frac{P(j2\pi f)}{Q(j2\pi f)} = \frac{\omega_o^2 j2\pi f}{\omega_o^2 + 2\omega_o j2\pi f + (j2\pi f)^2}, \qquad f \in \mathbb{R},$$

of a differential system $Q(D)y = P(D)u$, whose denominator and numerator polynomials are

$$Q(\lambda) = \omega_o^2 + 2\omega_o\lambda + \lambda^2, \qquad P(\lambda) = \omega_o^2\lambda.$$

Identifying $q_0 = \omega_o^2$, $q_1 = 2\omega_o$, $p_0 = p_2 = 0$ and $p_1 = \omega_o^2$, by 5.3.3 the system may be realized in state form as

$$\dot{x}(t) = \begin{bmatrix} -2\omega_o & 1 \\ -\omega_o^2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} \omega_o^2 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = [1 \quad 0]x(t), \qquad t \in \mathbb{R}.$$

A block diagram of the realization is given in Fig. 5.17. ∎

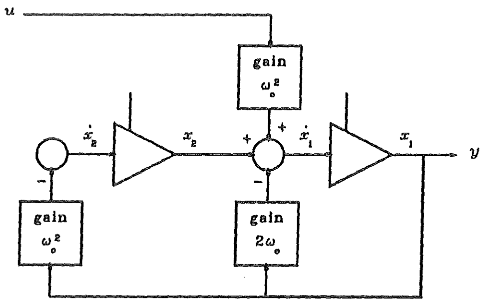


Figure 5.17. Realization of the approximate differentiator.

SOLUTION OF STATE EQUATIONS

In Section 5.2 we introduced state systems, and in Section 5.3 we showed how such systems may be implemented. In the present section we discuss the existence of solutions to state difference and differential equations, and their numerical solution.

The starting point for the description of discrete- and continuous-time state systems, with finite-dimensional state space $X = \mathbb{R}^N$ or $\mathbb{C}^N$ and finite-dimensional input and output signal ranges $U = \mathbb{R}^K$ or $\mathbb{C}^K$ and $Y = \mathbb{R}^M$ or $\mathbb{C}^M$, consists of their state difference or differential equation, respectively, and output equation:

Discrete-time: For $n \in \mathbb{Z}$

$$x(n+1) = f(n, x(n), u(n)),$$
$$y(n) = g(n, x(n), u(n)).$$

Continuous-time: For $t \in \mathbb{R}$

$$\dot{x}(t) = f(t, x(t), u(t)),$$
$$y(t) = g(t, x(t), u(t)).$$

Existence of Solutions

In the discrete-time case the solution of the state difference equation is clear. Given an initial time n_o, an initial state $x(n_o)$ and an input $u(n)$ for $n \geq n_o$, $n \in \mathbb{Z}$, the state difference equation

$$x(n+1) = f(n, x(n), u(n)), \qquad n \geq n_o, \ n \in \mathbb{Z},$$

may be solved for $x(n_o + 1)$, $x(n_o + 2)$, $\cdots$, by *successive substitution*. It is easy to program a digital computer to do this numerically, and sometimes it may be done analytically. Once the state trajectory $x(n)$, $n \geq n_o$, $n \in \mathbb{Z}$, has been found, it is straightforward to determine the corresponding output y using the output equation

$$y(n) = g(n, x(n), u(n)), \qquad n \geq n_o, \qquad n \in \mathbb{Z}.$$

For linear time-invariant state difference systems much can be inferred about the solution of the state difference equation before resorting to the computer. This is discussed in Section 5.5.

In the continuous-time case the situation is not quite so simple. First of all, the state differential equation does not always have a unique solution for a given initial state $x(t_o)$. Second, even if it does, it is not as easy as in the discrete-time case to compute the state trajectory. In the remainder of this section we first discuss the existence of solutions of the state differential equation, and then review some basic numerical methods for solving it.

The following theorem gives a well-known sufficient condition for the existence of unique solutions of a set of differential equations of the form $\dot{x}(t) = f(t, x(t))$, $t \in \mathbb{R}$. State differential equations are of this type when the input is a fixed given time function.

5.4.1. Summary: Existence of solutions to state differential equations. Suppose that the map $f: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following conditions:

(a) f is continuous on the set

$$Q = \{(t, x) \in \mathbb{R} \times \mathbb{R}^N \mid \|x - x_o\| \leq \alpha, |t - t_o| \leq \beta\},$$

with α and β some positive real constants, and

$$\|f(t, x)\| < \gamma \text{ for all } (t, x) \in Q$$

for some positive real constant γ;

(b) f satisfies the *Lipschitz condition*

$$\|f(t, x_1) - f(t, x_2)\| \leq \kappa\|x_1 - x_2\| \qquad \text{for all } (t, x_1), (t, x_2) \in Q$$

for some positive real constant κ.

Then if $\eta = \min(\alpha, \beta/\gamma)$ there exists a unique continuous function x defined on $[t_o - \eta, t_o + \eta]$ such that

(i) $x(t_o) = x_o$ and
(ii) x satisfies the differential equation $\dot{x}(t) = f(t, x(t))$ for $t \in [t_o - \eta, t_o + \eta]$. ∎

Rudolf Lipschitz (1832–1903) was a German mathematician who is well-known for his work on differential equations and Fourier series.

The norm $\|\cdot\|$ denotes any natural norm on $\mathbb{R}^N$.

5.4.2. Example: Nonunique solution. The standard example to show that if the conditions of 5.4.1 are not satisfied the initial value problem may have no unique solution is the scalar initial value problem

$$\dot{x}(t) = \sqrt{|x(t)|}, \qquad x(0) = 0.$$

The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(t, x) = \sqrt{|x|}$ is continuous everywhere but does not satisfy a Lipschitz condition on any set that includes the origin. Both

$$x(t) = 0, \qquad t \in \mathbb{R},$$

and

$$x(t) = \begin{cases} -\frac{1}{4}t^2 & \text{for } t < 0, \\ \frac{1}{4}t^2 & \text{for } t \geq 0, \end{cases} \qquad t \in \mathbb{R},$$

solve the initial value problem, which therefore has no unique solution. ∎

Numerical Integration of State Differential Equations

Even if the solution of a state differential equation is known to exist, it may be not easy or even impossible to find a closed-form solution. In this case numerical solution is called for. There exist many efficient methods for the numerical solution of

differential equations. We describe some of these for the solution of initial val problems of the form $\dot{x}(t) = f(x(t))$, $x(0) = x_o$. Initial value problems for tim varying differential equations of the form $\dot{x}(t) = f(t, x(t))$ may easily be reduced t this form.

5.4.3. Exercise: Reduction of a time-varying to a time-invariant equation Show that by defining an additional state variable $x_{N+1}(t) = t$ the initial valu problem

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_o) = x_o,$$

is equivalent to the initial value problem

$$\dot{z}(\tau) = g(z(\tau)), \qquad z(0) = z_o,$$

where $\tau = t - t_o$, and

$$z(\tau) = \begin{bmatrix} x(\tau) \\ x_{N+1}(\tau) \end{bmatrix}, \qquad g(z) = \begin{bmatrix} f(x_{N+1}, x) \\ 1 \end{bmatrix}, \qquad z_o = \begin{bmatrix} x_o \\ 0 \end{bmatrix}.$$ ∎

The simplest but usually not the most practical technique for solving the initial value problem $\dot{x} = f(x)$, $x(0) = x_o$, is *Euler's method*.

The Swiss mathematician Leonhard Euler (1707–1783) lived in Berlin and St. Petersburg. He was the central figure in the mathematical activities of the 18th century and the most prolific mathematician of all time.

5.4.4. Summary: Euler's method for approximating the solution of differential equations. Euler's method for approximating the solution of the initial value problem $\dot{x} = f(x)$, $x(0) = x_o$ consists of solving the approximating difference equation

$$x(t + T) = x(t) + T \cdot f(x(t)), \qquad t = 0, T, 2T, \cdots,$$

and interpolating linearly between the points $x(0), x(T), x(2T), \cdots$ thus found according to

$$x(t) = x(iT) + \frac{t - iT}{T} x((i + 1)T), \qquad iT \leq t \leq (i + 1)T,$$

$i = 0, 1, 2, \cdots$. If f satisfies the conditions of 5.4.1, as $T \to 0$ the function x thus obtained uniformly approaches the solution of the initial value problem on $[0, \eta]$. ∎

The number $T > 0$ is known as the *step size* of the integration method. Uniform convergence of the solution on $[0, \eta]$ means that the *largest* deviation between the approximate solution and the exact solution on this interval converges to zero as T approaches 0.

The idea behind the method is clear: Given any intermediate solution point $x(iT)$, compute the rate of change $\dot{x}(iT) = f(x(iT))$, and estimate the solution at an interval of length T later as $x((i+1)T) = x(iT) + T\cdot\dot{x}(iT)$.

In practice for many problems the step size T in Euler's method has to be chosen very small to obtain sufficiently accurate solutions. There are other methods that do better and require less computational effort for most problems. A scheme that is well known for its simplicity and reliability is that of *Runge-Kutta*. The method exists in infinitely many variants but we only describe the three most common ones. The Runge-Kutta order 1 scheme is in fact Euler's method.

5.4.5. Summary: Runge-Kutta order 1, 2, and 4 schemes for the integration of differential equations. Consider the initial value problem $\dot{x} = f(x)$ with $x(0)$ given. An approximate solution at the time points $0, T, 2T, \cdots$ is obtained as follows.

(a) The *first-order Runge-Kutta method* for obtaining an approximate solution x at time $(i+1)T$, given an approximate solution x_o at time iT, is to compute successively

$$r = f(x_o),$$
$$x = x_o + Tr.$$

(b) The *standard second-order Runge-Kutta method* for obtaining an approximate solution x at time $(i+1)T$, given an approximate solution x_o at time iT, is to compute successively

$$r_1 = f(x_o),$$
$$x_1 = x_o + \tfrac{1}{2}Tr_1, \qquad r_2 = f(x_1),$$
$$x = x_o + Tr_2.$$

(c) The *standard fourth-order Runge-Kutta method* for obtaining an approximate solution x at time $(i+1)T$, given an approximate solution x_o at time iT, is to compute successively

$$r_1 = f(x_o),$$
$$x_1 = x_o + \tfrac{1}{2}Tr_1, \qquad r_2 = f(x_1),$$
$$x_2 = x_o + \tfrac{1}{2}Tr_2, \qquad r_3 = f(x_2),$$
$$x_3 = x_o + Tr_3, \qquad r_4 = f(x_3),$$
$$x = x_o + \tfrac{1}{6}T(r_1 + 2r_2 + 2r_3 + r_4).$$

■

Carl Runge (1856–1927) and Wilhelm Kutta (1867–1944) both were German applied mathematicians.

The idea of the second-order method is first to compute the rate of change $\dot{x} = r_1$ at the given initial point, to use this to estimate the solution at mid-interval, and to obtain from this in turn a better estimate r_2 of the rate of change over the entire interval. The fourth-order scheme is a refinement of this procedure. There is of course a rationale for it, which may be found in the literature.

We present two examples.

5.4.6. Examples: Numerical solution of differential equations.

(a) *First-order equation.* By way of illustration we consider the numerical solution of the simple initial value problem

$$\dot{x}(t) = x(t), \qquad x(0) = 1.$$

It is clear that the solution is $x(t) = e^t$, $t \in \mathbb{R}$. In Table 5.1 the exact solution on the interval [0, 5] is compared with the numerical solution obtained by the Runge-Kutta schemes of orders 1, 2 and 4 with step size $T = 0.5$. This step size actually is quite large, because the relative change of the solution over one step is

$$\left|\frac{e^{t+T} - e^t}{e^t}\right| = e^T - 1 = 0.4687.$$

For this large step size Euler's method gives poor results, while the second-order Runge-Kutta scheme is better but also inaccurate. The fourth-order scheme yields a solution that is accurate to almost three decimal places on the interval [0, 5]. By taking a smaller step size much more accurate solutions may be obtained.

(b) *Moon rocket.* If we account for the expulsion of mass, the moon rocket of Example 5.2.2 is described by the state differential equation

$$\dot{x}_1(t) = x_2(t)/x_3(t),$$
$$\dot{x}_2(t) = cu(t) - gx_3(t),$$
$$\dot{x}_3(t) = -u(t).$$

We wish to solve these equations for $t \geq 0$ with the following data:

$$c = 2000 \text{ m/s}, \qquad x_1(0) = 1000 \text{ m},$$
$$g = 1.7 \text{ m/s}^2, \qquad x_2(0) = 0 \text{ kg}\cdot\text{m/s},$$
$$u(t) = 0.5 \text{ kg/s for } t \geq 0, \qquad x_3(0) = 1000 \text{ kg}.$$

TABLE 5.1 NUMERICAL SOLUTION OF THE DIFFERENTIAL EQUATION $\dot{x}(t) = x(t)$, $x(0) = 1$ ACCORDING TO THE RUNGE-KUTTA ORDER 1, 2, AND 4 SCHEMES WITH STEP SIZE $T = 0.5$

	x			
t	Order 1 (Euler)	Order 2	Order 4	Exact solution
0	1	1	1	1
0.5	1.5	1.625	1.648	1.649
1	2.25	2.641	2.717	2.718
1.5	3.375	4.291	4.479	4.482
2	5.063	6.973	7.384	7.389
2.5	7.594	11.33	12.17	12.18
3	11.39	18.41	20.06	20.09
3.5	17.09	29.92	33.08	33.12
4	25.63	48.62	54.42	54.60
4.5	38.44	79.01	89.88	90.02
5	57.67	128.4	148.2	148.4

Figure 5.18 shows the result of integrating the differential equation according to the standard second-order Runge-Kutta scheme with step size $T = 1$ s. Halving the step size results in a relative change of the various outcomes of less than 10^{-6} so that the accuracy seems sufficient.

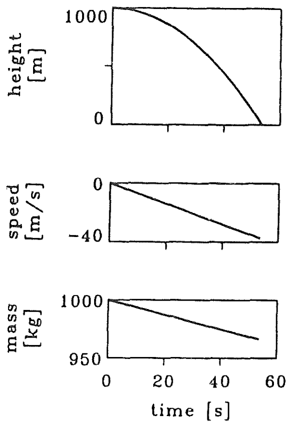


Figure 5.18. Flight trajectory of the moon rocket. Top: elevation. Middle: speed. Bottom: mass.

The initial state corresponds to the rocket being momentarily stationary at 1000 m above the moon surface with zero velocity. The thrust needed to keep the rocket stationary follows from $0 = cu - gx_3$ and would be provided by a constant mass expulsion rate of $u = gx_3/c = 1.7 \times 1000/2000 = 0.85$ kg/s. Because the actual mass expulsion rate u is less than this, the rocket starts descending. After almost 54 s the rocket crashes into the surface of the moon with a speed of nearly 37.5 m/s. Its initial mass of 1000 kg meanwhile has decreased to about 973 kg. ■

5.5 SOLUTION OF LINEAR STATE EQUATIONS

Quite explicit results may be obtained for the solution of the state equations of t following *linear* finite-dimensional state difference and differential systems:

Discrete-time

$$x(n+1) = A(n)x(n) + B(n)u(n),$$
$$y(n) = C(n)x(n) + D(n)u(n),$$

for $n \in \mathbb{T}$.

Continuous-time

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

for $t \in \mathbb{T}$.

In both cases, the time axis $\mathbb{T}$ may be right semi-infinite, such as $\mathbb{Z}_+$ or $\mathbb{R}_+$, (infinite.

For mathematical convenience we allow the input, state and output of linea state systems to be *complex-valued,* so that the state space is $X = \mathbb{C}^N$, while the inpu and output signal ranges are $U = \mathbb{C}^K$ and $Y = \mathbb{C}^M$, respectively. A is an $N \times N$ ma trix with time-dependent complex or real elements, B an $N \times K$ time-dependen matrix, C an $M \times N$ time-dependent matrix, and D, finally, an $M \times K$ time dependent matrix.

It is assumed in the following that in the continuous-time case the entries o the matrix functions A, B, C, and D are bounded and continuous on any finite time interval. These conditions ensure the existence of the solutions of the state differential and output equations.

Homogeneous State Difference and Differential Equations

We begin by studying the solution of the *homogeneous* state difference equation

$$x(n+1) = A(n)x(n), \qquad n \in \mathbb{T},$$

and the homogeneous state differential equation

$$\dot{x}(t) = A(t)x(t), \qquad t \in \mathbb{T}.$$

First consider the discrete-time case. If $n_o \in \mathbb{T}$ is some initial time, the solution of the homogeneous equation for $n \geq n_o$ is fully determined by the initial condition $x(n_o)$. Indeed by successive substitution it easily follows that

$$x(n) = A(n-1)A(n-2)\cdots A(n_o)x(n_o), \qquad n = n_o+1, n_o+2, \cdots.$$

We write this more compactly as

$$x(n) = \Phi(n, n_o)x(n_o), \qquad n = n_o+1, n_o+2, \cdots. \tag{1}$$

where Φ is the $N \times N$ matrix given by $\Phi(n, n_o) = A(n-1)A(n-2)\cdots A(n_o)$, for $n = n_o + 1, n_o + 2, \cdots$. The matrix Φ is called the *state transition matrix* of the system, because it describes the transition of the state from time n_o to time n (for the homogeneous system.)

The relation (1) shows that the transition from the initial state $x(n_o)$ to the state $x(n)$ at the fixed time n is a *linear* map. Indeed, this follows directly from the linearity of the state difference equation.

Also in the continuous-time case the map from the initial state $x(t_o)$, with t_o some initial time, to the state $x(t)$, with t some other time, is linear. We thus may write

$$x(t) = \Phi(t, t_o)x(t_o), \qquad t, t_o \in \mathbb{T},$$

where again the matrix function Φ is called the *state transition matrix* of the continuous-time system. By 5.4.1 the solution of continuous-time initial value problems, if it exists, also exists for time instants *before* the time instant t_o. Hence, in contrast to the discrete-time case, the continuous-time transition matrix Φ generally is also defined for $t < t_o$.

The continuous-time transition matrix is less easy to find than for the discrete-time case. Before turning to this problem we summarize the results found so far.

5.5.1. Summary: Solution of the homogeneous state difference and differential equation.

The solution of the homogeneous state difference equation

$$x(n+1) = A(n)x(n),$$

with $n \geq n_o$ and n and n_o in $\mathbb{T}$, may be expressed as

$$x(n) = \Phi(n, n_o)x(n_o), \qquad n \geq n_o.$$

The $N \times N$ *transition matrix* $\Phi(n, n_o)$ is defined for all n and n_o in $\mathbb{T}$ such that $n \geq n_o$.

The solution of the homogeneous state differential equation

$$\dot{x}(t) = A(t)x(t),$$

$t \in \mathbb{T}$, may be expressed as

$$x(t) = \Phi(t, t_o)x(t_o)$$

for all and t and t_o in $\mathbb{T}$, where the $N \times N$ *transition matrix* $\Phi(t, t_o)$ is defined for all t and t_o in $\mathbb{T}$. ∎

State Transition Matrix

We continue with a discussion of the state transition matrix Φ. As seen previously, in the discrete-time case we have an explicit expression for Φ:

5.5.2. Summary: Discrete-time transition matrix. The transition matrix of the homogeneous state difference equation $x(n+1) = A(n)x(n)$, $n \in \mathbb{T}$, is given by

$$\Phi(n, n_o) = \begin{cases} I & \text{for } n = n_o, \\ A(n-1)A(n-2)\cdots A(n_o) & \text{for } n > n_o, \end{cases} \qquad n, n_o \in \mathbb{T}. \quad ∎$$

In the continuous-time case such an explicit expression generally is not available, except if $N = 1$ (i.e., the state is one-dimensional so that the state differential equation $\dot{x} = Ax$ reduces to a *scalar* equation).

5.5.3. Exercise: Continuous-time transition matrix for the scalar case. Prove that if the state differential equation $\dot{x}(t) = A(t)x(t)$, $t \in \mathbb{T}$, is a *scalar* equation, the transition matrix may be expressed as

$$\Phi(t, t_o) = \exp\left(\int_{t_o}^{t} A(\tau)\,d\tau\right), \qquad t, t_o \in \mathbb{T}.$$

Hint: Use separation of variables to solve the differential equation $dx(t)/dt = A(t)x(t)$, $t \in \mathbb{T}$. ∎

Both in the discrete-time and the continuous-time case the transition matrix has a number of important properties, which may be summarized as follows.

5.5.4. Summary: Properties of the transition matrix.

(a) The transition matrix Φ of a state difference system satisfies the matrix difference equation

$$\Phi(n+1, n_o) = A(n)\Phi(n, n_o),$$

for $n \geq n_o$ with n and n_o in $\mathbb{T}$, with the initial condition

$$\Phi(n_o, n_o) = I.$$

(b) Φ has the *consistency property*

$$\Phi(n, n) = I \qquad \text{for all} \quad n \in \mathbb{T}.$$

(c) Φ possesses the *semigroup property*

$$\Phi(n_2, n_1)\Phi(n_1, n_o) = \Phi(n_2, n_o)$$

for all n_o, n_1 and n_2 in $\mathbb{T}$ such that $n_o \leq n_1 \leq n_2$.

(a′) The transition matrix of a state differential system satisfies the matrix differential equation

$$\frac{\partial}{\partial t}\Phi(t, t_o) = A(t)\Phi(t, t_o)$$

for all t and t_o in $\mathbb{T}$, with the initial condition

$$\Phi(t_o, t_o) = I.$$

(b′) Φ has the *consistency property*

$$\Phi(t, t) = I \qquad \text{for all} \quad t \in \mathbb{T}.$$

(c′) Φ possesses the *semigroup property*

$$\Phi(t_2, t_1)\Phi(t_1, t_o) = \Phi(t_2, t_o)$$

for all t_o, t_1 and t_2 in $\mathbb{T}$.

(d′) In particular, $\Phi(t, t_o)$ is nonsingular for every t and t_o in $\mathbb{R}$, and

$$\Phi^{-1}(t, t_o) = \Phi(t_o, t). \quad ∎$$

5.5.5. Proof. We indicate the proof for the continuous-time case only. That for the discrete-time case is similar.

The proof of (a′) follows by substituting $x(t) = \Phi(t, t_o)x(t_o)$ into the state differential equation. It follows that $[\partial\Phi(t, t_o)/\partial t]x(t_o) = A(t)\Phi(t, t_o)x(t_o)$. Since this holds identical in $x(t_o)$, the differential equation for Φ immediately follows. The initial condition $\Phi(t_o, t_o) = I$ is obtained by substituting $t = t_o$ into $x(t) = \Phi(t, t_o)x(t_o)$. This at the same time yields the consistency property (b′).

The semigroup property (c′) is proved by writing $x(t_2) = \Phi(t_2, t_1)x(t_1) = \Phi(t_2, t_1)\Phi(t_1, t_o)x(t_o)$, while, on the other hand, $x(t_2) = \Phi(t_2, t_o)x(t_o)$. The property follows because $\Phi(t_2, t_o)x(t_o) = \Phi(t_2, t_1)\Phi(t_1, t_o)x(t_o)$ holds identical in $x(t_o)$.

The inversion formula of (d′), finally, follows from (c′) by setting $t_2 = t_o$. ∎

From 5.5.4(a) and (a′) we may give the following interpretation of the transition matrix Φ: The ith column of Φ is the solution of the homogeneous state difference or differential equation for the initial condition col$(0, \cdots, 0, 1, 0, \cdots, 0)$, with the 1 in the ith position. In the continuous-time case this provides a way to find the transition matrix numerically by solution of the homogeneous state differential equation with appropriate initial conditions.

In the continuous-time case the semigroup property holds for any t_o, t_1 and t_2, while in the discrete-time case we need the ordering $n_o \le n_1 \le n_2$. Because of this, and because moreover the discrete-time transition matrix $\Phi(n, n_o)$ is not defined for $n < n_o$, the inversion formula (d′) has no discrete-time equivalent.

By way of example we consider two first-order systems.

5.5.6. Examples: Transition matrices.

(a) *Exponential smoother*. In 5.3.2(a) we found that the exponential smoother may be represented by the state difference and output equations

$$
\begin{aligned}
x(n+1) &= ax(n) + a(1-a)u(n),\\
y(n) &= x(n) + (1-a)u(n), \qquad n \in \mathbb{Z}.
\end{aligned}
$$

The homogeneous state difference equation is

$$x(n+1) = ax(n), \qquad n \in \mathbb{Z}.$$

It follows that the 1×1 state transition matrix is given by

$$
\begin{aligned}
\Phi(n, n_o) &= A(n-1)A(n-2)\cdots A(n_o) = a\cdot a\cdots a\\
&= a^{n-n_o}, \qquad n \ge n_o, \qquad n, n_o \in \mathbb{Z}.
\end{aligned}
$$

The consistency and semigroup properties are easily seen to hold.

(b) *RC network*. In 5.2.7 we found that the RC network may be described by the state differential equation

$$\dot{x}(t) = -\frac{1}{RC}x(t) + \frac{1}{R}u(t), \qquad t \in \mathbb{R},$$

and output equation

$$y_1(t) = \frac{1}{C}x(t), \qquad t \in \mathbb{R},$$

where the state x is the charge of the capacitor. The homogeneous state differential equation is

$$\dot{x}(t) = -\frac{1}{RC}x(t), \qquad t \in \mathbb{R}.$$

Given the initial condition $x(t_o)$, the solution of the homogeneous equation is

$$x(t) = e^{-\frac{t-t_o}{RC}}x(t_o), \qquad t \in \mathbb{R}.$$

It follows that the state transition matrix is

$$\Phi(t, t_o) = e^{-\frac{t-t_o}{RC}}, \qquad t, t_o \in \mathbb{R}.$$

Also here it is simple to see that the consistency and semigroup properties hold. ∎

Transition Matrix of Time-Invariant Systems

We now consider the important case that the state difference or differential system besides linear is also time-invariant. This results in quite explicit formulas for the state transition matrix.

5.5.7. Summary: Transition matrix of linear time-invariant state difference and differential systems.

The state transition matrix of the homogeneous linear time-invariant finite-dimensional state difference equation

$$x(n+1) = Ax(n), \quad n \ge n_o, \quad n \in \mathbb{T},$$

with A a constant matrix, is given by

$$\Phi(n, n_o) = A^{n-n_o},$$

The state transition matrix of the homogeneous linear time-invariant finite-dimensional state differential equation

$$\dot{x}(t) = Ax(t), \qquad t \in \mathbb{T},$$

with A a constant matrix, is given by

$$\Phi(t, t_o) = e^{A(t-t_o)},$$

for $n \geq n_o$, with n and n_o in $\mathbb{T}$. By definition, $A^0 = I$.

for all t and t_o in $\mathbb{T}$. ∎

For every square matrix M the *exponential matrix* e^M is defined by the converging infinite sum

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots.$$

The result 5.5.7 shows that in the discrete-time case the transition matrix of a time-invariant system simply follows by taking suitable powers of the matrix A. In the continuous-time case the transition matrix follows as the exponential of the matrix $A(t - t_o)$. The exponential of a matrix is defined as an infinite matrix power series whose coefficients are those of the exponential power series. Before showing some examples we discuss the proof of 5.5.7.

5.5.8. Proof of 5.5.7.

Discrete-time case. In the discrete-time case the transition matrix follows immediately from 5.5.2.

Continuous-time case. The proof for the continuous-time relies on *Picard's algorithm* for the solution of differential equations, which may be stated as follows: Given the differential equation

$$\dot{x}(t) = f(t, x(t)), \qquad t \in \mathbb{R},$$

with the initial condition $x(t_o) = x_o$, construct the sequence of functions $x_{(k)}(t)$, $t_o - \eta \leq t \leq t_o + \eta$, for $k = 0, 1, 2, \cdots$, as follows:

$$x_{(0)}(t) = x_o,$$

$$x_{(k+1)}(t) = x_o + \int_{t_o}^{t} f(\tau, x_{(k)}(\tau))\, d\tau, \qquad t_o - \eta \leq t \leq t_o + \eta,$$

for $k = 0, 1, 2, \cdots$. Then if f satisfies the conditions of 5.4.1 the sequence $x_{(k)}$, $k = 0, 1, 2, \cdots$, converges uniformly to the solution of the initial value problem $\dot{x}(t) = f(t, x(t))$, $t \in [t_o - \eta, t_o + \eta]$, $x(t_o) = x_o$.

The differential equation $\dot{x}(t) = Ax(t)$, $t \in \mathbb{R}$, satisfies the existence condition of 5.4.1 on any time interval. We find successively

$$x_{(0)}(t) = x_o,$$

$$x_{(1)}(t) = x_o + \int_{t_o}^{t} Ax_o\, d\tau = x_o + A(t - t_o)x_o,$$

$$\begin{aligned} x_{(2)}(t) &= x_o + \int_{t_o}^{t} A[x_o + A(\tau - t_o)x_o]\, d\tau \\ &= x_o + A(t - t_o)x_o + \frac{1}{2!}A^2(t - t_o)^2 x_o. \end{aligned}$$

By induction it is easily proved that in general

$$x_{(k)}(t) = \sum_{j=0}^{k} \frac{1}{j!}A^j(t - t_o)^j x_o, \qquad k = 0, 1, 2, \cdots, \qquad t \in \mathbb{R}.$$

Because the sequence $x_{(k)}$ converges uniformly to the solution x of the initial value problem as $k \to \infty$ for any x_o, the infinite matrix sum

$$\sum_{j=0}^{\infty} \frac{1}{j!}A^j(t - t_o)^j$$

converges for every $t - t_o$ and every A, and hence equals the state transition matrix $\Phi(t, t_o)$. The coefficients of the infinite matrix sum are precisely the coefficients of the power series for the exponential function, so that we denote this sum as $e^{A(t-t_o)}$. It follows that

$$\Phi(t, t_o) = e^{A(t-t_o)}.$$

We consider a simple example.

5.5.9. Examples: Transition matrix of linear time-invariant state differential systems.

(a) *RC network.* In 5.2.7 we obtained the state differential equation

$$\dot{x}(t) = -\frac{1}{RC}x(t) + \frac{1}{R}u(t), \qquad t \in \mathbb{R},$$

for the RC network. Since $A = -1/RC$, it follows that

$$\begin{aligned} e^{At} &= 1 - \frac{t}{RC} + \frac{1}{2!}\left(\frac{t}{RC}\right)^2 - \frac{1}{3!}\left(\frac{t}{RC}\right)^3 + \cdots \\ &= e^{-\frac{t}{RC}}, \qquad t \in \mathbb{R}. \end{aligned}$$

As a result, the transition matrix (which in this case is 1×1) is

$$\Phi(t, t_o) = e^{A(t-t_o)} = e^{-\frac{t-t_o}{RC}}, \qquad t, t_o \in \mathbb{R}.$$

This agrees with what we found in 5.5.6(b).

(b) *Moon rocket*. In 5.2.2 we saw that if the mass of the moon rocket is assumed to be *constant*, the system is described by the state differential equation

$$\dot{x}_1(t) = x_2(t),$$

$$\dot{x}_2(t) = \frac{c}{m}u(t) - g,$$

while the output equation is

$$y(t) = x_1(t).$$

Because of the presence of the term $-g$ in the second component of the state differential equation the moon rocket is not precisely a linear time-invariant differential system. This may be remedied by not taking the mass flow u as input, but the difference

$$w = u - \frac{mg}{c}.$$

The quantity mg/c indicates the size of the mass flow whose thrust compensates gravity. The state differential equation now becomes $\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = cw(t)/m$, so that in matrix form we have

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ c/m \end{bmatrix} w(t),$$

$$y(t) = [1 \quad 0]x(t).$$

Since

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

it follows that $A^j = 0$ for $j = 2, 3, \cdots$, so that

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \qquad t \in \mathbb{R}.$$

As a result, the transition matrix of the system is

$$\Phi(t, t_o) = \begin{bmatrix} 1 & t - t_o \\ 0 & 1 \end{bmatrix}, \qquad t, t_o \in \mathbb{R}.$$

In the last example, fortuitously the infinite matrix sum for the exponential e^{t} reduces to a finite sum, consisting of only two terms. Normally this does not happen. In Section 5.6 it is seen how by *diagonalization* of the matrix A convenien expressions may be obtained for the transition matrix, both in the discrete- and th continuous-time case.

Solution of the Inhomogeneous Equation

Once the transition matrix Φ of a linear state difference or differential system has been found, the solution of the *inhomogeneous* equation may also be obtained.

5.5.10. Summary: Solution of the inhomogeneous state difference and differential equation.

The solution of the state difference equation

$$x(n + 1) = A(n)x(n) + B(n)u(n),$$

$n \in \mathbb{T}$, is

$$x(n) = \Phi(n, n_o)x(n_o) + \sum_{k=n_o}^{n-1} \Phi(n, k + 1)B(k)u(k),$$

for $n \geq n_o$ with n and n_o in $\mathbb{Z}$. If the lower limit of the sum exceeds the upper limit the sum is canceled.

The solution of the state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

$t \in \mathbb{T}$, is

$$x(t) = \Phi(t, t_o)x(t_o) + \int_{t_o}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau,$$

for any t and t_o in $\mathbb{T}$. ■

5.5.11. Proof.

(a) *Discrete-time case*. The proof of the discrete-time solution follows by induction on n.

(b) *Continuous-time case*. The continuous-time solution is usually proved by the "variation of constant" argument. Given that any solution of the *homogeneous* equation has the form $x(t) = \Phi(t, t_o)a$, with a a suitable constant vector, we attempt to find a solution of the *inhomogeneous* equation of the form $x(t) = \Phi(t, t_o)a(t)$, with $a(t)$, $t \in \mathbb{R}$, a time-varying vector to be determined. Note that $a(t_o) = x(t_o)$. Differentiation of $x(t) = \Phi(t, t_o)a(t)$ with respect to time yields by application of the product rule

$$\dot{x}(t) = \left[\frac{\partial}{\partial t}\Phi(t, t_o)\right]a(t) + \Phi(t, t_o)\dot{a}(t) = A(t)\Phi(t, t_o)a(t) + \Phi(t, t_o)\dot{a}(t).$$

Substitution into the nonhomogeneous state differential equation results in $A(t)\Phi(t, t_o)a(t) + \Phi(t, t_o)\dot{a}(t) = A(t)\Phi(t, t_o)a(t) + B(t)u(t)$, which shows that

$$\Phi(t, t_o)\dot{a}(t) = B(t)u(t), \qquad t \in \mathbb{R}.$$

Using the inversion formula for the transition matrix (see 5.5.4(d')) we have $\dot{a}(t) = \Phi(t_o, t)B(t)u(t)$, which after integration from t_o results in

$$\begin{aligned} a(t) &= a(t_o) + \int_{t_o}^{t} \Phi(t_o, \tau)B(\tau)u(\tau)\, d\tau \\ &= x(t_o) + \int_{t_o}^{t} \Phi(t_o, \tau)B(\tau)u(\tau)\, d\tau, \qquad t \in \mathbb{R}. \end{aligned}$$

Substitution into $x(t) = \Phi(t, t_o)a(t)$ finally yields with the use of the semi-group property (see 5.5.4(c')) that

$$\begin{aligned} x(t) &= \Phi(t, t_o)\left[x(t_o) + \int_{t_o}^{t} \Phi(t_o, \tau)B(\tau)u(\tau)\, d\tau\right] \\ &= \Phi(t, t_o)x(t_o) + \int_{t_o}^{t} \Phi(t, t_o)\Phi(t_o, \tau)B(\tau)u(\tau)\, d\tau \\ &= \Phi(t, t_o)x(t_o) + \int_{t_o}^{t} \Phi(t, \tau)B(\tau)u(\tau)\, d\tau, \qquad t \in \mathbb{R}, \end{aligned}$$

which is what we set out to prove. ∎

We elaborate a little on the output y that results from the solution of the inhomogeneous state difference or differential equation. With the state expressed as in 5.5.10, it immediately follows with the output equation that in the discrete-time case the output of the system is

$$y(n) = C(n)\Phi(n, n_o)x(n_o) + \sum_{k=n_o}^{n-1} C(n)\Phi(n, k+1)B(k)u(k) + D(n)u(n),$$

for $n \geq n_o$ with n and n_o in $\mathbb{Z}$. Similarly, in the continuous-time case

$$y(t) = C(t)\Phi(t, t_o)x(t_o) + \int_{t_o}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)\, d\tau + D(t)u(t),$$

for all t and t_o in $\mathbb{R}$. We may write both expressions in the form

$$y = y_{\text{zero-input}} + y_{\text{zero-state}},$$

where in the discrete-time case

$$\begin{aligned} y_{\text{zero-input}}(n) &= C(n)\Phi(n, n_o)x(n_o), \\ y_{\text{zero-state}}(n) &= \sum_{k=n_o}^{n-1} C(n)\Phi(n, k+1)B(k)u(k) + D(n)u(n), \end{aligned}$$

and in the continuous-time case

$$\begin{aligned} y_{\text{zero-input}}(t) &= C(t)\Phi(t, t_o)x(t_o), \\ y_{\text{zero-state}}(t) &= \int_{t_o}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)\, d\tau + D(t)u(t). \end{aligned}$$

The term $y_{\text{zero-input}}$ is called the *zero-input response,* because it is the response to the initial state if the input is zero. The term $y_{\text{zero-state}}$ constitutes the *zero-state response,* because it forms the response if the initial state is zero.

In the time-invariant case these expressions reduce to

$$\begin{aligned} y_{\text{zero-input}}(n) &= CA^{n-n_o}x(n_o), \\ y_{\text{zero-state}}(n) &= \sum_{k=n_o}^{n-1} CA^{n-k-1}Bu(k) + Du(n), \end{aligned}$$

in the discrete-time case, while in the continuous-time case

$$\begin{aligned} y_{\text{zero-input}}(t) &= Ce^{A(t-t_o)}x(t_o), \\ y_{\text{zero-state}}(t) &= \int_{t_o}^{t} Ce^{A(t-\tau)}Bu(\tau)\, d\tau + Du(t). \end{aligned}$$

The zero-state response may be rewritten as

$$y_{\text{zero-state}}(n) = \sum_{k=n_o}^{\infty} h(n-k)u(k),$$

in the discrete-time case, and in the continuous-time case as

$$y_{\text{zero-state}}(t) = \int_{t_o}^{\infty} h(t-\tau)u(\tau)\, d\tau,$$

where the *impulse response matrix h* is given by

$$h(n) = CA^{n-1}B\mathbb{1}(n-1) + D\Delta(n), \qquad n \in \mathbb{Z},$$

in the discrete-time case, and in the continuous-time case as

$$h(t) = Ce^{At}B\mathbb{1}(t) + D\delta(t), \qquad t \in \mathbb{R}.$$

If the input u and the output y are both scalar (i.e., $K = M = 1$), the impulse response h is a scalar function. It may be interpreted as the zero-state response to the unit pulse Δ in the discrete-time case and the delta function δ in the continuous-time case. In the multi-input multi-output case (i.e., K and M are greater than 1) the (i, j) element h_{ij} is the zero-state response of the ith component of the output when the jth component of the input is Δ or δ and the remaining components of the input are zero.

The following example illustrates how the solution to the inhomogeneous equation may be found if the state transition matrix is known.

5.5.12. Example: Moon rocket. In Example 5.5.9(b) we found that if the mass of the moon rocket is assumed to be constant, and its input is the difference $w = u - mg/c$ of the mass expulsion rate u and the rate mg/c that keeps the rocket stationary, the rocket may be described by the state differential equation $\dot{x} = Ax + Bw$, where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ c/m \end{bmatrix}.$$

Also in 5.5.9(b) we found that

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

It follows that the solution of the state differential equation with given initial state at time 0 is

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bw(\tau)\,d\tau \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) + \int_0^t \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ c/m \end{bmatrix} w(\tau)\,d\tau \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) + \frac{c}{m} \int_0^t \begin{bmatrix} t-\tau \\ 1 \end{bmatrix} w(\tau)\,d\tau, \qquad t \geq 0. \end{aligned}$$

Suppose as in Example 5.4.6(b), where we integrated the state differential equation of the rocket numerically, that the input w is constant, say $w(t) = w_o$ for $t \geq 0$. It easily follows that

$$\left.\begin{aligned} x_1(t) &= x_1(0) + x_2(0)t + \frac{c}{2m}w_o t^2, \\ x_2(t) &= x_2(0) + \frac{c}{m}w_o t, \end{aligned}\right\} \qquad t \geq 0.$$

With the numerical values as in 5.4.6(b), namely, $c/m = 2000/1000 = 2$ m/kg·s, $x_1(0) = 1000$ m, $x_2(0) = 0$ m/s, and $w_o = 0.5 - 0.85 = -0.35$ kg/s, it follows that

$$\left.\begin{aligned} x_1(t) &= 1000 - 0.35t^2 \text{ [m]}, \\ x_2(t) &= -0.7t \text{ [m/s]}, \end{aligned}\right\} \qquad t \geq 0.$$

The corresponding trajectories are plotted in Fig. 5.19. They are very similar to those of Fig. 5.18, where they were computed while accounting for the change of mass of the rocket during the flight. It follows from the present solution that the rocket crashes onto the surface of the moon at the time t_f given by

$$0 = 1000 - 0.35t_f^2,$$

so that $t_f = \sqrt{1000/0.35} = 53.45$ s. The velocity of the rocket at the time of impact is $v(t_f) = -0.7t_f = -37.44$ m/s. Because these calculations do not account for the change of mass of the rocket, the results are only approximate.

Suppose that we take the output of the system as

$$y(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

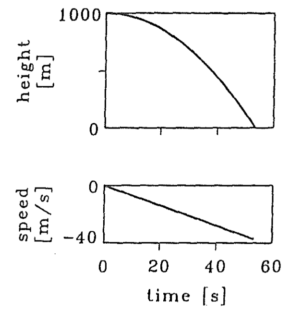


Figure 5.19. Flight trajectory of the moon rocket assuming constant mass. Top: elevation. Bottom: speed.

that is, the elevation x_1 is the first component of the output, and the speed x_2 the second component. Since $C = I$ and $D = 0$ the impulse response matrix h of the system is

$$h(t) = Ce^{At}B\mathbb{1}(t) + D\delta(t)$$
$$= \frac{c}{m}\begin{bmatrix} t \\ 1 \end{bmatrix}\mathbb{1}(t), \qquad t \in \mathbb{R}.$$

This shows that the zero-state response of the elevation to a delta function at time zero is a ramp, while that of the speed is a step. ∎

5.5.13. Review: Sampled state difference systems. Sampled linear state difference systems are described by a state difference and output equation of the form

$$x(t + T) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t), \qquad t \in \mathbb{T},$$

where the time axis $\mathbb{T}$ may be semi-infinite, such as $\mathbb{Z}_+(T)$, or $\mathbb{Z}(T)$. Given the initial state $x(t_o)$, the solution of the homogeneous state difference equation

$$x(t + 1) = A(t)x(t), \qquad t \in \mathbb{T},$$

may be written as

$$x(t) = \Phi(t, t_o)x(t_o), \qquad t \in \mathbb{T},$$

where the *transition matrix* Φ is given by

$$\Phi(t, t_o) = \begin{cases} I & \text{for } t = t_o, \\ A(t - T)A(t - 2T) \cdots A(t_o) & \text{for } t > t_o, \end{cases} \qquad t \in \mathbb{T}.$$

The transition matrix Φ has the following properties. First of all it satisfies the matrix difference equation

$$\Phi(t + T, t_o) = A(t)\Phi(t, t_o), \qquad t \geq t_o, \qquad t, t_o \in \mathbb{T},$$
$$\Phi(t_o, t_o) = I.$$

Besides the *consistency* property $\Phi(t, t) = I$ for all $t \in \mathbb{T}$ the transition matrix possesses the *semigroup* property

$$\Phi(t_2, t_1)\Phi(t_1, t_0) = \Phi(t_2, t_0)$$

for all t_2, t_1 and t_0 in $\mathbb{T}$ such that $t_2 \geq t_1 \geq t_0$. In the time-invariant case, i.e., when the matrix function A is a constant matrix,

$$\Phi(t, t_o) = A^{\frac{t-t_o}{T}}, \qquad t \geq t_o, \qquad t, t_o \in \mathbb{T}.$$

The solution of the inhomogeneous state difference equation may be expressed as

$$x(t) = \Phi(t, t_o)x(t_o) + \sum_{\substack{t_o \leq \tau < t, \\ \tau \in \mathbb{Z}(T)}} \Phi(t, \tau + T)B(\tau)u(\tau),$$

for $t \geq t_o$ and $t \in \mathbb{T}$. The zero-input and zero-state response of the system hence are

$$y_{\text{zero-input}}(t) = C(t)\Phi(t, t_o)x(t_o), \qquad t \geq t_o, \qquad t, t_o \in \mathbb{Z}(T),$$

$$y_{\text{zero-state}}(t) = T \sum_{\substack{\tau \geq t_o, \\ \tau \in \mathbb{Z}(T)}} k(t, \tau)u(\tau), \qquad t \geq t_o, \qquad t, t_o \in \mathbb{Z}(T),$$

where the kernel k is given by

$$k(t, \tau) = \begin{cases} \frac{1}{T}C(t)\Phi(t, \tau + T)B(\tau) & \text{for } t \geq \tau + T, \\ \frac{1}{T}D(t) & \text{for } t = \tau, \\ 0 & \text{for } t < \tau, \end{cases} \qquad t, \tau \in \mathbb{Z}(T).$$

In the time-invariant case the zero-state response is

$$y_{\text{zero-state}}(t) = T \sum_{\substack{\tau \geq t_o, \\ \tau \in \mathbb{Z}(T)}} h(t - \tau)u(\tau), \qquad t \geq t_o, \qquad t, t_o \in \mathbb{Z}(T),$$

where the impulse response h is given by

$$h(t) = \frac{1}{T}[CA^{\frac{t-T}{T}}B\mathbb{1}(t - T) + D\Delta(t)], \qquad t \in \mathbb{Z}(T).$$ ∎

5.6 MODAL ANALYSIS OF LINEAR TIME-INVARIANT STATE SYSTEMS

The state of a system describes the internal situation of the system. The choice of the state is not unique, however, in the sense that there may be many different descriptions of the same system, each with different state variables, that *externally* show the same behavior. Any nonsingular transformation of the state, for instance, is also a state.

In this section we consider state transformations for linear time-invariant difference and differential systems that bring the system in *modal* form. *Modes* are special motions of the system that are decoupled and facilitate the analysis of the dynamic behavior.

All discrete-time systems in this section are considered on the infinite time axis $\mathbb{Z}$ and all continuous-time systems on $\mathbb{R}$.

Time-Invariant State Transformations

First *linear time-invariant state transformations* are discussed. By writing the state as

$$x = Vx',$$

with V a nonsingular constant matrix and x' the transformed state, the state differential and output equations assume a different form, as summarized in the following.

5.6.1. Summary: Linear time-invariant state transformations.

Let V be a constant nonsingular $N \times N$ matrix. Then the transformation

$$x = Vx'$$

transforms the equations

$$x(n+1) = Ax(n) + Bu(n),$$
$$y(n) = Cx(n) + Du(n), \qquad n \in \mathbb{Z}.$$

into

$$x'(n+1) = A'x'(n) + B'u(n),$$
$$y(n) = C'x'(n) + D'u(n), \qquad n \in \mathbb{Z},$$

where

$$A' = V^{-1}AV, \qquad B' = V^{-1}B,$$
$$C' = CV, \qquad D' = D.$$

Let V be a constant nonsingular $N \times N$ matrix. Then the transformation

$$x = Vx'$$

transforms the equations

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t), \qquad t \in \mathbb{R},$$

into

$$\dot{x}'(t) = A'x'(t) + B'u(t),$$
$$y(t) = C'x'(t) + D'u(t), \qquad t \in \mathbb{R},$$

where

$$A' = V^{-1}AV, \qquad B' = V^{-1}B,$$
$$C' = CV, \qquad D' = D.$$

■

The proof is not difficult.

5.6.2. Proof. We consider the proof only for the discrete-time case. That f(the continuous-time case is very similar. Substitution of $x(n) = Vx'(n)$ into the sta difference equation $x(n+1) = Ax(n) + Bu(n)$ yields $Vx'(n+1) = AVx'(n)$ · $Bu(n)$, which, after premultiplication by V^{-1}, results in $x'(n+1)$ = $V^{-1}AVx'(n) + V^{-1}Bu(n) = A'x'(n) + B'u(n)$. For the output equation we hav $y(n) = Cx(n) + Du(n) = CVx'(n) + Du(n) = C'x'(n) + D'u(n)$, which com pletes the proof.

A question of interest is how the state transition matrix and impulse response matri: behave under state transformation.

5.6.3. Summary: Effect of state transformation on the state transition and impulse response matrices.

(a) The transition matrix Φ' of a discrete-time system that has been subjected to a state transformation $x = Vx'$ is related to the transition matrix Φ of the system before transformation as

$$\Phi'(n, n_o) = V^{-1}\Phi(n, n_o)V,$$

for all n and n_o in $\mathbb{Z}$ such that $n \geq n_o$.

(b) The impulse response matrix of the system remains unchanged under the state transformation.

(a') The transition matrix Φ' of a continuous-time system that has been subjected to a state transformation $x = Vx'$ is related to the transition matrix Φ of the system before transformation as

$$\Phi'(t, t_o) = V^{-1}\Phi(t, t_o)V,$$

for all t and t_o in $\mathbb{R}$.

(b') The impulse response matrix of the system remains unchanged under the transformation. ■

5.6.4. Proof.

(a) Again for the discrete-time case, if the input is zero we have $x'(n) = V^{-1}x(n) = V^{-1}\Phi(n, n_o)x(n_o) = V^{-1}\Phi(n, n_o)Vx'(n_o)$, which shows that the transformed system has the transition matrix $V^{-1}\Phi(n, n_o)V$.

(b) Because the zero state transforms into the zero state, the zero state response of the transformed system is the same as the zero state response of the system before transformation, which means that the impulse response matrix of the system is not changed by the state transformation. ■

It follows from 5.6.3 that the state transition matrix is subject to the same transformation as the A-matrix. The impulse response matrix is *not* changed by the state transformation. The impulse response is an *external* property of the system, which is not affected by its internal representation.

Modal Transformations

The particular state transformation that is of interest in this section is the *modal* transformation. This transformation is closely related to the eigenvectors and eigenvalues of the matrix A occurring in the state difference or differential equation.

We briefly review some notions from linear algebra. The complex number λ is an *eigenvalue* of the $N \times N$ matrix A if

$$\det(\lambda I - A) = 0,$$

with I the $N \times N$ unit matrix. The eigenvalues of A hence are the N roots of the polynomial $\det(\lambda I - A)$ of degree N, which is called the *characteristic polynomial* of the matrix A. The nonzero vector $v \in \mathbb{C}^N$ is an *eigenvector* of A corresponding to the eigenvalue λ if

$$Av = \lambda v.$$

If the N eigenvalues of A are all *different*, A always has N linearly independent eigenvectors. If some of the eigenvalues of A are equal to each other, A may or may not have N linearly independent eigenvectors. If it does not, A is called *defective*.

5.6.5. Exercise: Similarity transformation. The transformation of A to $A' = V^{-1}AV$ is called a *similarity transformation* of the matrix A. Prove that a similarity transformation does not affect the eigenvalues, i.e., A and A' have the same eigenvalues. ■

A modal transformation is a state transformation such that the columns of the transform matrix V are the eigenvectors of the matrix A. This is only possible if A is not defective. The case where A is defective is discussed in Supplement D. If A is not defective, modal transformation brings it into *diagonal* form. We summarize as follows.

5.6.6. Summary: Modal transformation. Suppose that A is an $N \times N$ matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$, and N corresponding linearly independent eigenvectors $v_1, v_2, \cdots, v_N$. Let V be the $N \times N$ matrix whose columns are the eigenvectors $v_1, v_2, \cdots, v_N$, that is,

$$V = [v_1 \quad v_2 \quad \cdots \quad v_N],$$

and let Λ be the $N \times N$ diagonal matrix whose diagonal elements are the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$, that is,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \lambda_N \end{bmatrix} = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N).$$

Then the state transformation

$$x = Vx'$$

results in the transformations

$$V^{-1}AV = \Lambda, \tag{1a}$$

$$V^{-1}A^iV = \Lambda^i \quad \text{for all} \quad i \in \mathbb{Z}_+, \tag{1b}$$

$$V^{-1}e^{At}V = e^{\Lambda t} \quad \text{for all} \quad t \in \mathbb{R}, \tag{1c}$$

where

$$\Lambda^i = \operatorname{diag}(\lambda_1^i, \lambda_2^i, \cdots, \lambda_N^i), \qquad i \in \mathbb{Z}_+,$$

$$e^{\Lambda t} = \operatorname{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_N t}), \qquad t \in \mathbb{R}.$$

Conversely,

$$A = V\Lambda V^{-1},$$

$$A^i = V\Lambda^i V^{-1} \quad \text{for all} \quad i \in \mathbb{Z}_+,$$

$$e^{At} = Ve^{\Lambda t}V^{-1} \quad \text{for all} \quad t \in \mathbb{R}.$$ ■

Note that if $\lambda = 0$ we define

$$\lambda^i = \begin{cases} 1 & \text{for } i = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad i \in \mathbb{Z}_+.$$

A matrix V such that $V^{-1}AV$ is diagonal is said to *diagonalize* the matrix A. Diagonalization of A is only possible if A is not defective, that is, if A has N linearly independent eigenvectors.

It follows from 5.6.6 that once A has been diagonalized, it is a simple matter to determine powers of A and its exponential, which is what we need if we wish to find the transition matrix of the system.

5.6.7. Proof of 5.6.6. By the definition of V it follows that

$$\begin{aligned} AV &= A[v_1 \quad v_2 \quad \cdots \quad v_N] = [Av_1 \quad Av_2 \quad \cdots \quad Av_N] \\ &= [\lambda_1 v_1 \quad \lambda_2 v_2 \quad \cdots \quad \lambda_N v_n] \\ &= [v_1 \quad v_2 \quad \cdots \quad v_N] \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \lambda_N \end{bmatrix} = V\Lambda. \end{aligned}$$

Because by assumption the eigenvectors $v_1, v_2, \cdots, v_N$ are linearly independent, the matrix V is nonsingular. Premultiplication of $AV = V\Lambda$ by V^{-1} yields $V^{-1}AV = \Lambda$, which proves (1a).

The equality (1b) follows by writing

$$\begin{aligned} \Lambda^i &= (V^{-1}AV)^i = V^{-1}AV \cdot V^{-1}AV \cdots V^{-1}AV = V^{-1}A \cdots AV \\ &= V^{-1}A^iV. \end{aligned}$$

To prove (1c), we write $e^{\Lambda t}$ in the form of an exponential series and use (1b). It follows that

$$\begin{aligned} e^{\Lambda t} &= I + \Lambda t + \frac{1}{2!}\Lambda^2 t^2 + \frac{1}{3!}\Lambda^3 t^3 + \cdots \\ &= I + V^{-1}AVt + \frac{1}{2!}V^{-1}A^2Vt^2 + \frac{1}{3!}V^{-1}A^3Vt^3 + \cdots \\ &= V^{-1}\left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right]V \\ &= V^{-1}e^{At}V, \qquad t \in \mathbb{R}. \end{aligned}$$

It is a simple matter to derive the remaining equalities of 5.6.6. ∎

Once a modal transformation of a discrete- or continuous-time system has been performed, the state transition matrix of the original system from the latter part of 5.6.6 may be obtained as

$$\Phi(n, n_o) = A^{n-n_o} = V\Lambda^{n-n_o}V^{-1}, \qquad n \ge n_o, \qquad n, n_o \in \mathbb{Z},$$

in the discrete-time case, and

$$\Phi(t, t_o) = e^{A(t-t_o)} = Ve^{\Lambda(t-t_o)}V^{-1}, \qquad t, t_o \in \mathbb{R},$$

in the continuous-time case. The following examples illustrate the modal transforma tion.

5.6.8. Examples: Modal transformation.

(a) *Second-order smoother.* In 5.3.5(a) we found that the second-orde smoother of 4.2.4(b) may be described by the state difference systen $x(n+1) = Ax(n) + Bu(n)$, $y(n) = Cx(n) + Du(n)$, $n \in \mathbb{Z}$, with

$$A = \begin{bmatrix} a_1 & 1 \\ a_o & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 + a_1 b_2 \\ a_0 b_2 \end{bmatrix},$$
$$C = [1 \quad 0], \qquad D = b_2.$$

Adopting the numerical values $a_0 = 0$, $a_1 = \frac{1}{2}$, $b_1 = 1$ and $b_2 = 0$ this reduces to

$$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$C = [1 \quad 0], \qquad D = 0.$$

The characteristic polynomial of the matrix A is

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - \frac{1}{2} & -1 \\ 0 & \lambda \end{bmatrix}\right) = \lambda(\lambda - \tfrac{1}{2}).$$

As a result, the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = \frac{1}{2}$. It is easily found that the corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

It follows that the modal transformation matrix V, its inverse V^{-1} and the diagonal matrix Λ are

$$V = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}, \qquad V^{-1} = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Thus, after modal transformation the system is represented as $x'(n+1) = A'x'(n) + B'u(n)$, $y(n) = C'x'(n) + D u(n)$, $n \in \mathbb{Z}$, where

$$A' = V^{-1}AV = \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad B' = V^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C' = CV = [1 \quad 1].$$

The state transition matrix of the transformed system is

$$\Phi'(n, n_o) = \Lambda^{n-n_o} = \begin{bmatrix} \Delta(n-n_o) & 0 \\ 0 & (\frac{1}{2})^{n-n_o} \end{bmatrix}, \qquad n \geq n_o.$$

It follows for the impulse response of the transformed system, which is also the impulse response of the untransformed system,

$$\begin{aligned} h(n) &= C'\Lambda^{n-1}B'\mathbb{1}(n-1) + D\Delta(n) = [1 \quad 1]\begin{bmatrix} \Delta(n-1) & 0 \\ 0 & (\frac{1}{2})^{n-1} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= (\tfrac{1}{2})^{n-1}\mathbb{1}(n-1), \qquad n \in \mathbb{Z}. \end{aligned}$$

We may also compute the transition matrix Φ of the untransformed system as

$$\begin{aligned} \Phi(n, n_o) &= A^{n-n_o} = V\Lambda^{n-n_o}V^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} \Delta(n-n_o) & 0 \\ 0 & (\frac{1}{2})^{n-n_o} \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} (\frac{1}{2})^{n-n_o} & -2\Delta(n-n_o) + 2(\frac{1}{2})^{n-n_o} \\ 0 & \Delta(n-n_o) \end{bmatrix}, \qquad n \geq n_o, \end{aligned}$$

with n and n_o both belonging to $\mathbb{Z}$. This may be simplified to $\Phi(n_o, n_o) = I$ and

$$\Phi(n, n_o) = \begin{bmatrix} (\frac{1}{2})^{n-n_o} & 2(\frac{1}{2})^{n-n_o} \\ 0 & 0 \end{bmatrix}, \qquad n > n_o.$$

(b) *RCL network.* In 5.2.1 we considered a series connection of a resistor, a capacitor and an inductor, which may be described by the state differential equation $\dot{x}(t) = Ax(t) + Bu(t)$, where

$$A = \begin{bmatrix} 0 & 1/L \\ -1/C & -R/L \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The characteristic polynomial of the matrix A is

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1/L \\ 1/C & \lambda + R/L \end{bmatrix}\right) = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}.$$

Assume that $R = 11\ \Omega$, $L = 0.01$ H and $C = 0.001$ F, so that the characteristic polynomial is $\lambda^2 + 1100\lambda + 100000 = (\lambda + 100)(\lambda + 1000)$. As a result, A has two eigenvalues given by $\lambda_1 = -100$ and $\lambda_2 = -1000$. It is easily found that two corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 100 \\ -1000 & -1100 \end{bmatrix}$$

are

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ -10 \end{bmatrix}.$$

It follows that the modal transformation matrix V, its inverse V^{-1} and the diagonal matrix Λ are

$$V = \begin{bmatrix} 1 & 1 \\ -1 & -10 \end{bmatrix}, \qquad V^{-1} = \frac{1}{9}\begin{bmatrix} 10 & 1 \\ -1 & -1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -100 & 0 \\ 0 & -1000 \end{bmatrix}.$$

Hence, the transition matrix Φ of the system *before* modal transformation is given by

$$\begin{aligned} \Phi(t, t_o) &= e^{A(t-t_o)} = Ve^{\Lambda(t-t_o)}V^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & -10 \end{bmatrix}\begin{bmatrix} e^{-100(t-t_o)} & 0 \\ 0 & e^{-1000(t-t_o)} \end{bmatrix}\frac{1}{9}\begin{bmatrix} 10 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{10}{9}e^{-100(t-t_o)} - \frac{1}{9}e^{-1000(t-t_o)} & \frac{1}{9}e^{-100(t-t_o)} - \frac{1}{9}e^{-1000(t-t_o)} \\ -\frac{10}{9}e^{-100(t-t_o)} + \frac{10}{9}e^{-1000(t-t_o)} & -\frac{1}{9}e^{-100(t-t_o)} + \frac{10}{9}e^{-1000(t-t_o)} \end{bmatrix}, \end{aligned}$$

for t and $t_o \in \mathbb{R}$. ■

Modes

By using the modal transform, the zero-input response of the state of the discrete-time system $x(n+1) = Ax(n) + Bu(n)$, $n \in \mathbb{Z}$, may be written as

$$x(n) = A^n x(0) = V\Lambda^n V^{-1}x(0), \qquad n = 0, 1, 2, \cdots.$$

We may expand this expression as follows. The columns of V are the eigenvectors $v_1, v_2, \cdots, v_N$ of A. Denote the *rows* of the inverse matrix V^{-1} as $w_1, w_2, \cdots, w_N$. Then we have

$$x(n) = [v_1\ v_2 \cdots v_N] \begin{bmatrix} \lambda_1^n & 0 & 0 & 0 & 0 \\ 0 & \lambda_2^n & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \lambda_N^n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_N \end{bmatrix} x(0)$$

$$= \sum_{i=1}^{N} v_i \lambda_i^n w_i x(0), \qquad n = 0, 1, \cdots.$$

Because w_i is a row vector and $x(0)$ a column vector (of the same dimension), the quantity $w_i x(0)$, which we denote as α_i, is a *scalar*. We thus may write

$$x(n) = \sum_{i=1}^{N} \alpha_i \lambda_i^n v_i, \qquad n = 0, 1, \cdots.$$

This shows that the zero-input response is a *linear combination* of the solutions

$$\lambda_i^n v_i, \qquad n = 0, 1, \cdots,$$

for $i = 1, 2, \cdots, N$. Each of these solutions is called a *mode*. Because

$$x(0) = \sum_{i=1}^{N} \alpha_i v_i,$$

the numbers α_i simply are the coefficients of the expansion of the initial state $x(0)$ in the vectors $v_1, v_2, \cdots, v_N$. A mode is said to be *excited* if the corresponding expansion coefficient α_i is nonzero.

In the continuous-time case we similarly have for the zero-input response

$$x(t) = e^{At}x(0) = Ve^{\Lambda t}V^{-1}x(0) = \sum_{i=1}^{N} v_i e^{\lambda_i t} w_i x(0)$$

$$= \sum_{i=1}^{N} \alpha_i e^{\lambda_i t} v_i, \qquad t \geq 0.$$

Here the modes are solutions of the form $e^{\lambda_i t} v_i$, $t \geq 0$.

We summarize as follows.

5.6.9. Summary: Modes of a linear time-invariant system.

Suppose that the $N \times N$ matrix A has the eigenvalues

$$\lambda_1, \lambda_2, \cdots, \lambda_N,$$

and N corresponding linearly independent eigenvectors

$$v_1, v_2, \cdots, v_N.$$

Then the solution of the homogeneous state difference equation

$$x(n+1) = Ax(n), \qquad n \in \mathbb{Z}_+,$$

is a linear combination of the N modes

$$\lambda_i^n v_i, \qquad n \in \mathbb{Z}_+,$$

$i = 1, 2, \cdots, N$. The coefficient α_i of the ith mode is the ith component of the expansion of $x(0)$ in the vectors $v_1, v_2, \cdots, v_N$.

Suppose that the $N \times N$ matrix A has the eigenvalues

$$\lambda_1, \lambda_2, \cdots, \lambda_N,$$

and N corresponding linearly independen eigenvectors

$$v_1, v_2, \cdots, v_N.$$

Then the solution of the homogeneous state differential equation

$$\dot{x}(t) = Ax(t), \qquad t \in \mathbb{R}_+,$$

is a linear combination of the N modes

$$e^{\lambda_i t} v_i, \qquad t \in \mathbb{R}_+,$$

$i = 1, 2, \cdots, N$. The coefficient α_i of the ith mode is the ith component of the expansion of $x(0)$ in the vectors $v_1, v_2, \cdots, v_N$. ∎

The modal expansion of the zero-input response gives a good idea of the general behavior of the zero-input response. If the eigenvalue λ_i is *real*, for instance, also the corresponding eigenvector and hence the corresponding mode may be chosen real. The time behavior of the mode is in this case exponential. In the discrete-time case the mode *decreases exponentially* if $|\lambda_i| < 1$, *increases exponentially* if $|\lambda_i| > 1$, and remains *bounded* without decreasing to zero if $|\lambda_i| = 1$. In the continuous-time case, the mode decreases exponentially if $\lambda_i < 0$, increases exponentially if $\lambda_i > 0$, and remains bounded without decreasing to zero if $\lambda_i = 0$.

When A has *complex* eigenvalues, the corresponding modes are complex-valued functions of time. If for physical reasons the state of the system is essentially real-valued, it is useful to have a real representation for the complex modes.

5.6.10. Summary: Real representation of complex modes.

(a) Suppose that the real matrix A has a complex eigenvalue λ with corresponding eigenvector v. Then the complex conjugate $\bar{\lambda}$ of λ is also an eigenvalue of A and the complex conjugate $\bar{v}$ of v is a corresponding eigenvector.

(b) Let

$$\lambda = \rho e^{j\psi},$$

with $\rho = |\lambda|$ and ψ real, and

$$v = r + js,$$

(b′) Let

$$\lambda = \sigma + j\omega,$$

with σ and ω real, and

$$v = r + js,$$

with r and s real vectors. Then if the initial state $x(0)$ lies in the space spanned by r and s, the solution of the homogeneous state difference equation

$$x(n+1) = Ax(n), \qquad n \in \mathbb{Z}_+,$$

may be expressed as

$$x(n) = \alpha\rho^n[r\cos(\psi n + \phi) - s\sin(\psi n + \phi)]$$

$n \in \mathbb{Z}_+$, with α and ϕ real constants that are determined by $x(0)$.

with r and s real vectors. Then if the initial state $x(0)$ lies in the space spanned by r and s, the solution of the homogeneous state differential equation

$$\dot{x}(t) = Ax(t), \qquad t \in \mathbb{R}_+,$$

may be expressed as

$$x(t) = \alpha e^{\sigma t}[r\cos(\omega t + \phi) - s\sin(\omega t + \phi)],$$

$t \in \mathbb{R}_+$, with α and ϕ real constants that are determined by $x(0)$. ∎

5.6.11. Proof.

(a) This fact is well-known from linear algebra.

(b) Since $x(0)$ lies in the space spanned by r and s, there exist real constants c_1 and c_2 such that $x(0) = c_1 r + c_2 s = \frac{1}{2}(\mu v + \overline{\mu v})$, where $\mu = c_1 - jc_2$. Because v and $\bar{v}$ are eigenvectors of A corresponding to the eigenvalues λ and $\bar{\lambda}$, respectively, the associated solution of the homogeneous difference equation is

$$x(n) = \tfrac{1}{2}(\mu\lambda^n v + \bar{\mu}\bar{\lambda}^n \bar{v}), \qquad n \in \mathbb{Z}_+.$$

Substitution of $\lambda = \rho e^{j\psi}$, $v = r + js$ and $\mu = \alpha e^{j\phi}$, with α and ϕ suitable real numbers, completes the proof.

(b′) The proof of the continuous-time result is similar. ∎

We see from 5.6.10 that the modes corresponding to a complex conjugate pair of eigenvalues λ, $\bar{\lambda}$ may be combined to a real solution that contains two arbitrary constants.

In the discrete-time case, this solution is *harmonically damped* if the magnitude $\rho = |\lambda|$ of the eigenvalue is less than 1, *purely harmonic* if $\rho = 1$, and *harmonically increasing* if $\rho > 1$. The number $\psi = \arg(\lambda)$ is the *angular frequency* of the harmonic solution.

In the continuous-time case, the solution is harmonically damped if the real part $\sigma = \mathrm{Re}(\lambda)$ of the eigenvalue is less than 0, purely harmonic if $\sigma = 0$, and harmonically increasing if $\sigma > 0$. The imaginary part $\omega = \mathrm{Im}(\lambda)$ is the angular frequency of the harmonic.

We consider two examples.

5.6.12. Examples: Modes.

(a) *Second-order smoother.* In Example 5.6.8(a) we found that the second-order smoother has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \frac{1}{2}$, with corresponding eigen-

vectors $v_1 = \mathrm{col}(1, -\frac{1}{2})$ and $v_2 = \mathrm{col}(1, 0)$. As a result, the system has the two real modes

$$\lambda_1^n v_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}\Delta(n), \qquad \lambda_2^n v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tfrac{1}{2})^n, \qquad n \in \mathbb{Z}_+.$$

Any solution of the homogeneous equation on $\mathbb{Z}_+$ is a linear combination of these modes. The first of the modes is what is called a "dead-beat" mode, because any initial component along this mode reduces exactly to zero after a finite number of time instants (in this case after one time instant). The second mode decays exponentially. It decreases quickly, because after, say, five time instants this mode reduces to $(\frac{1}{2})^5 = 1/32$ of its initial value.

(b) *Fourth-order RCL network.* As a continuous-time example we consider the electrical network of Fig. 5.20. It contains a current source, which provides the input u to the system, two capacitors, two inductors, and two resistors. For the resistors, capacitors, and inductors we may write the following ten element equations:

Resistors:

$$v_{R_1} = R_1 i_{R_1}, \qquad v_{R_2} = R_2 i_{R_2}.$$

Capacitors:

$$q_{C_1} = C_1 v_{C_1}, \qquad \dot{q}_{C_1} = i_{C_1}, \qquad q_{C_2} = C_2 v_{C_2}, \qquad \dot{q}_{C_2} = i_{C_2}.$$

Inductors:

$$\phi_{L_1} = L_1 i_{L_1}, \qquad \dot{\phi}_{L_1} = v_{L_1}, \qquad \phi_{L_2} = L_2 i_{L_2}, \qquad \dot{\phi}_{L_2} = v_{L_2}.$$

In these equations v denotes voltage, i current, q charge, and ϕ flux, while the subscripts refer to the various circuit elements. The directions in which the currents are taken positive are indicated in the diagram. Also the current i in the connecting

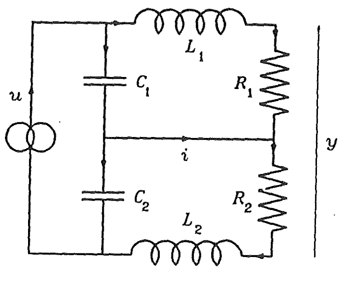


Figure 5.20. A fourth-order RCL network.

branch is shown. Application of Kirchhoff's voltage law to each of the three meshes yields successively

$$v_{C_1} = v_{L_1} + v_{R_1}, \qquad v_{C_2} = v_{R_2} + v_{L_2}, \qquad y = v_{R_1} + v_{R_2}.$$

Application of Kirchhoff's current law to each of the five nodes yields

$$u = i_{C_1} + i_{L_1}, \qquad i_{L_1} = i_{R_1}, \qquad i_{C_1} = i + i_{C_2}, \qquad i_{R_1} + i = i_{R_2}, \qquad i_{R_2} = i_{L_2}.$$

Because the charges and the fluxes determine the amount of energy that is stored in the network, the four state variables are ϕ_{L_1}, q_{C_1}, ϕ_{L_2} and q_{C_2}. The six element voltages (excluding the output voltage y) and the seven currents may be eliminated from the 18 equations. Defining the state as

$$x = \begin{bmatrix} \phi_{L_1} \\ q_{C_1} \\ \phi_{L_2} \\ q_{C_2} \end{bmatrix},$$

this leads to the state differential equation $\dot{x}(t) = Ax(t) + Bu(t)$ and output equation $y(t) = Cx(t)$, where

$$A = \begin{bmatrix} -R_1/L_1 & 1/C_1 & 0 & 0 \\ -1/L_1 & 0 & 0 & 0 \\ 0 & 0 & -R_2/L_2 & 1/C_2 \\ 0 & 0 & -1/L_2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

$$B = [R_1/L_1 \quad 0 \quad R_2/L_2 \quad 0].$$

To perform a modal analysis we first determine the eigenvectors and eigenvalues of the matrix A. The characteristic polynomial of A is easily found to be

$$\det(\lambda I - A) = \left(\lambda^2 + \frac{R_1}{L_1}\lambda + \frac{1}{L_1C_1}\right)\left(\lambda^2 + \frac{R_2}{L_2}\lambda + \frac{1}{L_2C_2}\right).$$

Suppose that $R_1 = 1$, $C_1 = 1$, $L_1 = 1/2$, $R_2 = 1$, $C_2 = 5/6$, and $L_2 = 1/5$. Then the characteristic polynomial is $(\lambda^2 + 2\lambda + 2)(\lambda^2 + 5\lambda + 6)$, whose roots are

$$\lambda_{1,2} = -1 \pm j, \qquad \lambda_3 = -2, \qquad \lambda_4 = -3.$$

It is not difficult to find that

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -5 & 0 \end{bmatrix}$$

has the corresponding eigenvectors

$$v_{1,2} = \begin{bmatrix} \frac{1}{2} \pm (-\frac{1}{2}j) \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 5 \end{bmatrix}, \qquad v_4 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 5 \end{bmatrix}.$$

For the eigenvalue pair $\lambda_{1,2} = -1 \pm j$ and corresponding eigenvector pair $v_{1,2}$ we have

$$\sigma_1 = \operatorname{Re}(\lambda_1) = -1, \qquad \omega_1 = \operatorname{Im}(\lambda_1) = 1,$$

$$r_1 = \operatorname{Re}(v_1) = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad s_1 = \operatorname{Im}(v_1) = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, according to 5.6.10 corresponding to the eigenvalue pair $\lambda_{1,2} = -1 \pm j$ the homogeneous state differential equation has the solution

$$\alpha e^{-t}\left(\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} \cos(t + \phi) - \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \sin(t + \phi)\right)$$

for $t \geq 0$, with α and ϕ arbitrary real constants. This solution represents damped harmonics with angular frequency 1. Corresponding to the eigenvalues $\lambda_3 = -2$ and $\lambda_4 = -3$ the equation has the solutions

$$\alpha_3 e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 2 \\ 5 \end{bmatrix}, \qquad \alpha_4 e^{-3t}\begin{bmatrix} 0 \\ 0 \\ 3 \\ 5 \end{bmatrix},$$

for $t \geq 0$, with α_3 and α_4 arbitrary real constants. Both solutions decay exponentially, the second faster than the first, and both faster than the harmonically damped solutions. The first solution pair corresponds to excitation of the top RCL mesh in the circuit, and the second to excitation of the bottom mesh. ■

5.6.13. Review: Modal analysis of sampled linear time-invariant state difference systems. We briefly discuss the modal analysis of sampled linear time-invariant state difference systems with the homogeneous state difference equation

$$x(t + T) = Ax(t), \qquad t \in \mathbb{Z}(T).$$

If A has the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$, with corresponding linearly independent eigenvectors $v_1, v_2, \cdots, v_N$, we may define the modal transformation matrix V and the diagonal matrix Λ as in 5.6.6. It follows for the transition matrix of the system

$$\Phi(t, t_o) = A^{\frac{t-t_o}{T}} = V\Lambda^{\frac{t-t_o}{T}}V^{-1}, \qquad t \geq t_o, \qquad t, t_o \in \mathbb{Z}(T).$$

The modes of the system are the solutions

$$\lambda_i^{t/T}v_i, \qquad t \in \mathbb{Z}_+(T), \qquad i = 1, 2, \cdots, N.$$

If λ is a complex eigenvalue with corresponding complex eigenvector v, we may write

$$\lambda = \rho e^{j\psi T}, \qquad v = r + js,$$

with $\rho = |\lambda|$ and ψ, r and s real. Then corresponding to the eigenvalue pair λ, $\bar{\lambda}$ the homogeneous state difference equation has the real solution

$$x(t) = \alpha\rho^{t/T}[r\cos(\psi t + \phi) - s\sin(\psi t + \phi)], \qquad t \in \mathbb{Z}_+(T). \qquad ■$$

STABILITY OF STATE SYSTEMS

In Section 4.6 we discussed the *stability* of input-output systems. Two types of stability were distinguished: *BIBO* and *CICO* stability. These notions also apply to state systems:

(a) *BIBO stability:* Bounded-Input Bounded-Output stability of a state system implies that if the input u of the system is bounded, any corresponding output y is also bounded from any finite time on.

(b) *CICO stability:* Converging-Input Converging-Output stability of a state system signifies that if two inputs u_1 and u_2 converge to each other as time increases, so do any two corresponding outputs y_1 and y_2.

In the case of state systems, a situation of particular interest arises when in these definitions the output is replaced with the state. This leads to the notion of *BIBS* (bounded-input bounded-state) stability, and that of CICS (converging-input converging-state) stability.

BIBS stability implies that in particular the zero-input response remains bounded. This is closely related to the notion of *Lyapunov stability*, which is extensively discussed in many more specialized texts. CICS stability implies that in particular the zero-input response converges to zero. This is connected to the idea of *asymptotic stability*, found in these same texts.

The discussion in this section is restricted to linear time-invariant state difference and differential systems.

Boundedness and Convergence of the Zero-Input State Response

Before entering a more complete discussion of the stability of state systems we consider the asymptotic behavior of the zero-input state response. The conditions for boundedness and convergence exhibit considerable similarity to those for difference and differential IO systems as given in 4.6.5.

5.7.1. Summary: Boundedness and convergence of the zero-input state response.

(a) Necessary and sufficient conditions for any solution of the homogeneous state difference equation

$$x(n+1) = Ax(n), \qquad n \in \mathbb{Z},$$

to remain bounded from any finite time on are that

(i) all eigenvalues of A have magnitude less than or equal to 1, and

(ii) to any eigenvalue with magnitude 1 and multiplicity m there correspond m linearly independent eigenvectors.

(b) Necessary and sufficient conditions for any solution of the homogeneous state difference equation to converge to zero as time increases are that all the eigenvalues of A have magnitude strictly less than 1.

(a′) Necessary and sufficient conditions for any solution of the homogeneous state differential equation

$$\dot{x}(t) = Ax(t), \qquad t \in \mathbb{R},$$

to remain bounded from any finite time on are that

(i′) all eigenvalues of A have real part less than or equal to 0, and

(ii′) to any eigenvalue with real part 0 and multiplicity m there correspond m linearly independent eigenvectors.

(b′) Necessary and sufficient conditions for any solution of the homogeneous state differential equation to converge to zero as time increases are that all the eigenvalues of A have strictly negative real part. ■

If A is nondefective, the proof follows by inspection of the modal expansion of the zero-input response, as discussed in Section 5.6. The proof when A is defective relies on the Jordan normal form discussed in Supplement E.

We illustrate the result with a simple example.

5.7.2. Example: Boundedness and convergence. Consider the state differential equation $\dot{x} = Ax$, with

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A has the eigenvalues -2, -3, 0, and 0. Corresponding to the double eigenvalue 0 the matrix A has the two linearly independent eigenvectors col(0, 0, 1, 0) and col(0, 0, 0, 1). A thus satisfies the conditions of 5.7.1(a') but not those of (b'). Hence, all solutions are bounded but they do not all converge to zero. Indeed, the solution of the state differential equation is easily seen to be given by

$$\begin{aligned} x_1(t) &= e^{-2t}x_1(0), \\ x_2(t) &= e^{-3t}x_2(0), \\ x_3(t) &= x_3(0), \\ x_4(t) &= x_4(0), \end{aligned} \qquad t \geq 0.$$

Inspection shows that each of the components of x and hence x itself is bounded. The last two components do not converge to 0, however.

Suppose now that A is modified to

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues still are -2, -3, 0, and 0, but the matrix only has a single eigenvector col(0, 0, 1, 0) corresponding to the double eigenvalue 0. As a result, the conditions of neither 5.7.1(a') nor (b') are satisfied, and, hence, not all solutions are bounded and converge to zero. Indeed, it may easily be verified that the solution is given by

$$\begin{aligned} x_1(t) &= e^{-2t}x_1(0), \\ x_2(t) &= e^{-3t}x_2(0), \\ x_3(t) &= x_3(0) + x_4(0)t, \\ x_4(t) &= x_4(0), \end{aligned} \qquad t \geq 0.$$

The third component of x, and, hence, x itself, is unbounded when $x_4(0) \neq 0$. ■

BIBO, CICO, BIBS, and CICS Stability of State Systems

We limit ourselves to a rather simple-minded discussion of the stability of state systems. The full story covering all contingencies is quite intricate. The solution of the state difference equation

$$x(n + 1) = Ax(n) + Bu(n), \qquad n \in \mathbb{Z}_+, \tag{1}$$

may be written as

$$x(n) = A^n x(0) + \sum_{k=0}^{n} h(n - k)u(k), \qquad n \in \mathbb{Z}_+,$$

where

$$h(n) = A^{n-1}B\mathbb{1}(n - 1), \qquad n \in \mathbb{Z}.$$

The matrix function h is the impulse response matrix of the system when the state taken as the output. Suppose that the eigenvalues of the matrix A all have magnitud strictly less than 1. Then,

(i) by 5.7.1 the zero-input response $A^n x(0)$, $n \in \mathbb{Z}_+$, is bounded and converges ex ponentially to zero, and
(ii) each of the entries of the impulse response matrix h has finite action, so that by 4.6.9 the zero-state response to any bounded input is bounded, and by 4.6.1: the response to any input that converges to zero also converges to zero.

It follows that the response of the system (1) to any bounded input is bounded, sc that the system is BIBS stable. Moreover, by linearity (ii) implies that the responses to any two inputs that converge to each other also converge to each other. Hence, the system is also CICS stable.

This shows that the condition that all eigenvalues have magnitude strictly less than 1 is *sufficient* for BIBS and CICS stability. By 5.7.1(b) the condition is also *necessary* for CICS stability, but not for BIBS stability (take, for instance, the system $x(n + 1) = x(n)$, $n \in \mathbb{Z}$).

Suppose now that the state difference equation (1) is complemented with an output equation

$$y(n) = Cx(n) + Du(n), \qquad n \in \mathbb{Z}.$$

Clearly, if the system is BIBS stable it is also BIBO stable, and if it is CICS stable it is also CICO stable. Hence, the condition that all eigenvalues of A have magnitude strictly less than 1 is not only sufficient for BIBS and CICS stability, but also for both BIBO and CICO stability. It is necessary for neither (think of the system $x(n + 1) = 2x(n)$, $n \in \mathbb{Z}$ with the trivial output equation $y(n) = 0$, $n \in \mathbb{Z}$).

Similar considerations apply to state differential systems. They lead to the conclusion that the condition that all eigenvalues of A have strictly negative real part is sufficient for BIBS, BIBO and CICO stability and necessary and sufficient for CICS stability.

We summarize our findings as follows.

5.7.3. Summary: BIBO, CICO, BIBS, and CICS stability of linear time-invariant state difference and differential systems.

A sufficient condition for the state difference system	A sufficient condition for the state differential system
$x(n + 1) = Ax(n) + Bu(n)$, $y(n) = Cx(n) + Du(n)$, $\quad n \in \mathbb{Z}$,	$\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$, $\quad t \in \mathbb{R}$,

to be BIBO, CICO, and BIBS stable is that all the eigenvalues of the matrix A have magnitude strictly less than one. The condition is necessary and sufficient for CICS stability.

to be BIBO, CICO, and BIBS stable is that all the eigenvalues of the matrix A have strictly negative real part. The condition is necessary and sufficient for CICS stability. ∎

In many applications, CICS stability is the preferred form of stability. The condition given in 5.7.3 is both necessary and sufficient for CICS stability.

The following examples illustrate the results.

5.7.4. Examples: Stability of state systems.

(a) *RCL network.* In Example 5.6.8(b) we found that for the assumed numerical values the eigenvalues of the RCL network are -100 and -1000. Hence, by 5.7.3 the network is BIBO, CICO, BIBS, and CICS stable.

(b) *A digital filter.* We show an example of a system for which the sufficient condition of 5.7.3 is not satisfied, but which still is both BIBO and CICO stable though not BIBS and CICS stable. This demonstrates that the sufficient condition is not always necessary.

Consider a second-order digital filter described by the state difference and output equations

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \rho \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = [1 \quad 0], \qquad D = 0,$$

with ρ a real number to be chosen. The eigenvalues of the matrix A are $\frac{1}{2}$ and ρ. If the magnitude $|\rho|$ of the second eigenvalue is not strictly less than 1, the sufficient condition of 5.7.3 is not satisfied.

However, the zero-input response of the system is given by

$$\begin{aligned} y_{\text{zero-input}}(n) &= CA^n x(0) = [1 \quad 0] \begin{bmatrix} (\frac{1}{2})^n & 0 \\ 0 & \rho^n \end{bmatrix} x(0) \\ &= (\tfrac{1}{2})^n x_1(0), \qquad n \in \mathbb{Z}_+, \end{aligned}$$

with x_1 the first component of the state x. Clearly, the zero-input response is bounded and converges to zero for any initial state. Furthermore, the impulse response h of the system is

$$\begin{aligned} h(n) &= CA^{n-1}B\mathbb{1}(n-1) + D\Delta(n) = [1 \quad 0] \begin{bmatrix} (\frac{1}{2})^{n-1} & 0 \\ 0 & \rho^{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbb{1}(n-1) \\ &= (\tfrac{1}{2})^{n-1}\mathbb{1}(n-1), \qquad n \in \mathbb{Z}. \end{aligned}$$

Because the impulse response has finite action, the zero-state response of the system to any bounded input is bounded and that to any input that converges to zero also converges to zero.

As a result, the system is both BIBO and CICO stable, even if the second eigenvalue does not have magnitude strictly less than 1. The reason is that by the particular structure of the matrices A, B, and C the eigenvalue ρ, which potentially may result in a nonconverging zero-input response and make the action of the impulse response infinite, in fact appears neither in the zero-input response nor in the impulse response.

The realization of the system of Fig. 5.21 clarifies the situation. The system consists of two separate subsystems, the second of which is not affected by the input and does not affect the output. The input-output stability of the overall system only depends on the stability of the first subsystem.

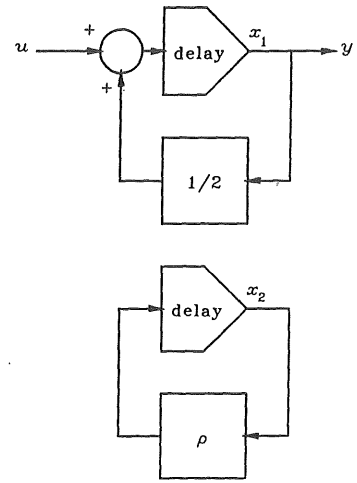


Figure 5.21. Realization of the digital filter.

On the other hand, it is clear that if $|\rho|$ is greater than 1, the system is both BIBS and CICS unstable. If the initial value of the second component x_2 of the state is nonzero, this component becomes larger with every time step and very soon overflow occurs.

The phenomenon that a particular eigenvalue of the system may not affect the zero-input response or the impulse response, or both, explains why certain systems do not satisfy the sufficient condition of 5.7.3 and still are BIBO stable or even CICO stable.

(c) *Moon rocket.* In Example 5.4.6(b) we saw that a constant input to the moon rocket causes both its elevation and its speed to increase without bound. Hence, the rocket is not BIBS stable.

To see that the rocket neither is CICS stable, consider two initial states such that the initial elevations are different but the initial speeds and masses are the same. Then if the input is zero, in the resulting free drops of the rocket the difference between the elevations remains constant and never diminishes. ∎

5.7.5. Review: Sampled state difference systems. The various stability definitions apply without modification to sampled state difference systems. A sufficient condition for the linear time-invariant sampled state difference system with state difference and output equation

$$x(t + T) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t), \qquad t \in \mathbb{Z}(T),$$

to be BIBO, CICO, and BIBS stable is that the eigenvalues of A all have magnitude strictly less than one. This condition is sufficient *and* necessary for the system to be CICS stable. ∎

5.8 FREQUENCY RESPONSE OF STATE SYSTEMS

In Section 3.7 we found that if a continuous- or discrete-time linear time-invariant input-output mapping system has a harmonic signal as input, the corresponding output signal is again harmonic, with the same frequency. All the system does is to multiply the harmonic input by a scaling function, called the *frequency response function* of the system.

In the present section we study the response of linear time-invariant state difference and differential systems to harmonic inputs. To define their frequency response function we consider the zero-state response. According to Section 5.5, the zero-state response with initial time $n_o = -\infty$ of the state difference system

$$x(n + 1) = Ax(n) + Bu(n), \tag{1a}$$
$$y(n) = Cx(n) + Du(n), \qquad n \in \mathbb{Z}, \tag{1b}$$

is

$$y(n) = \sum_{k=-\infty}^{\infty} h(n - k)u(k), \qquad n \in \mathbb{Z}, \tag{2}$$

where h is the impulse response matrix

$$h(n) = CA^{n-1}B\mathbb{1}(n - 1) + D\Delta(n), \qquad n \in \mathbb{Z}.$$

The zero-state response with initial time $t_o = -\infty$ of the state differential system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3a}$$
$$y(t) = Cx(t) + Du(t), \qquad t \in \mathbb{R}, \tag{3b}$$

is

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)u(\tau)\, d\tau, \qquad t \in \mathbb{R},$$

where the impulse response matrix h is

$$h(t) = Ce^{At}B\mathbb{1}(t) + D\delta(t), \qquad t \in \mathbb{R}.$$

Because the input u and the output y of state difference or differential systems may be vector-valued signals, the notion of frequency response function need be generalized to that of *frequency response matrix*.

Frequency Response Matrix

To extend the idea of frequency response function to that of frequency response matrix, we suppose that the input to the state difference system (1) is the *vector-valued* harmonic signal

$$u(n) = u_o e^{j2\pi fn}, \qquad n \in \mathbb{Z}.$$

Here, $f \in \mathbb{R}$ is a real frequency and $u_o \in \mathbb{C}^K$ a constant vector whose dimension is that of the input. Then, by (2) the zero-state response to this harmonic input, if it exists, is

$$y(n) = \sum_{k=-\infty}^{\infty} h(n - k)u_o e^{j2\pi fk}, \qquad n \in \mathbb{Z}.$$

Replacement of the summation variable k by $n - m$ results in

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)u_o e^{j2\pi f(n-m)} = \left(\sum_{m=-\infty}^{\infty} h(m)e^{-j2\pi fm}\right)u_o e^{j2\pi fn}$$
$$= \hat{h}(j2\pi f)u_o e^{j2\pi fn}, \qquad n \in \mathbb{Z},$$

where the *frequency response matrix* $\hat{h}$ is a matrix function of the same dimensions as the impulse response matrix h, and is given by

$$\hat{h}(f) = \sum_{m=-\infty}^{\infty} h(m)e^{-j2\pi fm}, \qquad f \in \mathbb{R}.$$

From Section 5.7 we know that if all the eigenvalues of the system matrix A have magnitude strictly less than one, all the elements of the impulse response matrix h have finite action. This is a sufficient condition for the existence of the frequency response matrix $\hat{h}$ of the system.

The frequency response matrix of the linear time-invariant state differential system (3) follows by considering vector-valued harmonic inputs of the form

$$u(t) = u_o e^{j2\pi ft}, \qquad t \in \mathbb{R}.$$

The zero-state response to this input with initial time $t_o = -\infty$, if it exists, is again harmonic of the form

$$y(t) = \hat{h}(f)u_o e^{j2\pi ft}, \qquad t \in \mathbb{R},$$

where the frequency response matrix $\hat{h}$ has the same dimensions as the impulse response matrix h, and is given by

$$\hat{h}(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt, \qquad f \in \mathbb{R}.$$

A sufficient condition for the existence of the frequency response matrix is that all the eigenvalues of the system matrix A have strictly negative real parts.

We summarize the results found so far.

5.8.1. Summary. Frequency response of linear time-invariant state difference and differential systems.

Suppose that the eigenvalues of the system matrix A of the linear time-invariant state difference system

$$x(n+1) = Ax(n) + Bu(n),$$
$$y(n) = Cx(n) + Du(n), \qquad n \in \mathbb{Z},$$

all have magnitude strictly less than one.

Then, the system has a well-defined zero-state response to the harmonic input

$$u(n) = u_o e^{j2\pi fn}, \qquad n \in \mathbb{Z},$$

with $u_o \in \mathbb{C}^K$ a constant vector and f a constant real frequency, of the form

$$y(n) = \hat{h}(f)u_o e^{j2\pi fn}, \qquad n \in \mathbb{Z}.$$

The *frequency response matrix* $\hat{h}$ is given by

$$\hat{h}(f) = \sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn}, \qquad f \in \mathbb{R},$$

with h the impulse response matrix of the system.

Suppose that the eigenvalues of the system matrix A of the linear time-invariant state differential system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t), \qquad t \in \mathbb{R},$$

all have strictly negative real part.

Then, the system has a well-defined zero-state response to the harmonic input

$$u(t) = u_o e^{j2\pi ft}, \qquad t \in \mathbb{R},$$

with $u_o \in \mathbb{C}^K$ a constant vector and f a constant real frequency, of the form

$$y(t) = \hat{h}(f)u_o e^{j2\pi ft}, \qquad t \in \mathbb{R}.$$

The *frequency response matrix* $\hat{h}$ is given by

$$\hat{h}(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt, \qquad f \in \mathbb{R},$$

where h is the impulse response matrix of the system. ∎

To determine the frequency response matrix explicitly we observe that if the input is harmonic the state is likely also to be harmonic of the form

$$x(n) = x_o e^{j2\pi fn}, \qquad n \in \mathbb{Z},$$

with x_o a constant vector to be determined. Substitution of this conjectured particular solution into the state difference equation results in

$$x_o e^{j2\pi f(n+1)} = Ax_o e^{j2\pi fn} + Bu_o e^{j2\pi fn}, \qquad n \in \mathbb{Z},$$

which after cancellation of the common factor $e^{j2\pi fn}$ simplifies to

$$x_o e^{j2\pi f} = Ax_o + Bu_o.$$

This expression is independent of time. After rewriting it as

$$(e^{j2\pi f}I - A)x_o = Bu_o$$

with I the $N \times N$ unit matrix, we may solve for the unknown constant vector x_o as

$$x_o = (e^{j2\pi f}I - A)^{-1}Bu_o.$$

It may be shown that under the condition that all eigenvalues of A have magnitude strictly less than one the inverse matrix in this expression always exists. Substitution of the particular solution $x(n) = x_o e^{j2\pi fn}$, $n \in \mathbb{Z}$, into the output equation results in

$$\begin{aligned} y(n) &= (Cx_o + Du_o)e^{j2\pi fn} \\ &= [C(e^{j2\pi f}I - A)^{-1}B + D]u_o e^{j2\pi fn}, \qquad n \in \mathbb{Z}. \end{aligned}$$

This shows that the frequency response matrix $\hat{h}$ of the state difference system is given by

$$\hat{h}(f) = C(e^{j2\pi f}I - A)^{-1}B + D, \qquad f \in \mathbb{R}.$$

The frequency response matrix may thus directly be found from the coefficient matrices A, B, C, and D.

Analogously, by considering a particular solution of the state differential equation of the form $x(t) = x_o e^{j2\pi ft}$, $t \in \mathbb{R}$, it is easily found that the frequency response matrix of the linear time-invariant state differential system (3) may explicitly be expressed in the coefficient matrices as

$$\hat{h}(f) = C(j2\pi fI - A)^{-1}B + D, \qquad f \in \mathbb{R}.$$

We summarize as follows.

5.8.2. Summary: Frequency response matrix of linear time-invariant state difference and differential systems.

The frequency response matrix $\hat{h}$ of the linear time-invariant state difference system

$$x(n+1) = Ax(n) + Bu(n),$$
$$y(n) = Cx(n) + Du(n), \qquad n \in \mathbb{Z},$$

exists if the eigenvalues of A all have magnitude strictly less than one, and is given by

$$\hat{h}(f) = C(e^{j2\pi f}I - A)^{-1}B + D, \qquad f \in \mathbb{R}.$$

The frequency response matrix of the linear time-invariant state differential system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t), \qquad t \in \mathbb{R},$$

exists if the eigenvalues of the matrix A all have strictly negative real part, and is given by

$$\hat{h}(f) = C(j2\pi fI - A)^{-1}B + D, \qquad f \in \mathbb{R}.$$

■

For multi-input multi-output (MIMO) systems with K-dimensional input and M-dimensional output the frequency response matrix $\hat{h}$ is an $M \times K$ matrix function. The (i, j)th element $\hat{h}_{ij}$ of the frequency response matrix $\hat{h}$ may be interpreted as follows. Suppose that the jth component of the input is harmonic with frequency f and all other components of the input are zero. Then the response of the ith component of the output equals $\hat{h}_{ij}(f)$ multiplied by the jth component of the input. For single-input single-output (SISO) systems $\hat{h}$ is a 1×1 function and is called, as before, the frequency response *function* of the system.

5.8.3. Example: RCL network. In Example 5.2.1 it was found that if the state of the RCL network is chosen to consist of the charge q of the capacitor and the flux ϕ of the inductor, the system has the state differential equation $\dot{x}(t) = Ax(t) + Bu(t)$, $t \in \mathbb{R}$, with

$$A = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{R}{L} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Suppose now that we consider *two* outputs simultaneously. Taking the first compo-

nent y_1 of the output as the current through the network we have from 5.2.1

$$y_1(t) = \frac{1}{L}x_2(t), \qquad t \in \mathbb{R}.$$

If we choose as the second component y_2 of the output the voltage v_L across the i
ductor we have as seen in 5.2.1

$$y_2(t) = -\frac{1}{C}x_1(t) - \frac{R}{L}x_2(t) + u(t), \qquad t \in \mathbb{R}.$$

Hence, the output equation is

$$y(t) = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \qquad t \in \mathbb{R}.$$

This is an output equation of the form

$$y(t) = Dx(t) + Eu(t),$$

with

$$D = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{R}{L} \end{bmatrix}, \qquad E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We determine the frequency response matrix $\hat{h}$ of the system. Because the output is two-dimensional and the input one-dimensional, the frequency response matrix is a 2×1 matrix function. The characteristic polynomial of the matrix A is $\lambda^2 + (R/L)\lambda + 1/LC$. It may be verified that if R, C, and L are positive constants, the two roots always have strictly negative real part. Hence, by 5.8.2 the frequency response matrix is well defined. Temporarily writing s for $j2\pi f$, we have for the matrix $sI - A$ and its inverse:

$$sI - A = \begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & s + \frac{R}{L} \end{bmatrix}, \qquad (sI - A)^{-1} = \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \begin{bmatrix} s + \frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & s \end{bmatrix}.$$

It follows that

$$D(sI - A)^{-1}B + E = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{R}{L} \end{bmatrix} \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \begin{bmatrix} s + \frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \begin{bmatrix} \frac{s}{L} \\ s^2 \end{bmatrix}.$$

Back substituting $s = j2\pi f$ we thus find for the frequency response matrix of the system

$$\hat{h}(f) = \begin{bmatrix} \dfrac{\frac{j2\pi f}{L}}{(j2\pi f)^2 + \frac{R}{L}j2\pi f + \frac{1}{LC}} \\ \dfrac{(j2\pi f)^2}{(j2\pi f)^2 + \frac{R}{L}j2\pi f + \frac{1}{LC}} \end{bmatrix}, \quad f \in \mathbb{R}.$$

The frequency response matrix has two entries. The top entry is the frequency response function from the input to the current through the network. The bottom entry is that from the input to the voltage across the inductor. ■

5.8.4. Review: Frequency response of sampled systems. The zero-state response of the linear time-invariant sampled state difference system

$$x(t + T) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t), \quad t \in \mathbb{Z}(T),$$

to the harmonic input

$$u(t) = u_o e^{j2\pi ft}, \quad t \in \mathbb{Z}(T),$$

with f a real frequency, is well-defined if all the eigenvalues of A have magnitude strictly less than one. The response is given by

$$y(t) = \hat{h}(f)u_o e^{j2\pi ft}, \quad t \in \mathbb{Z}(T),$$

where the frequency response matrix $\hat{h}$ may be expressed in terms of the impulse response matrix h of the system (see 5.5.13) as

$$\hat{h}(f) = T \sum_{t \in \mathbb{Z}(T)} h(t)e^{-j2\pi ft}, \quad f \in \mathbb{R}.$$

The frequency response matrix may be given directly in terms of the coefficient matrices as

$$\hat{h}(f) = C(e^{j2\pi fT}I - A)^{-1}B + D, \quad f \in \mathbb{R}.$$

■

5.9 PROBLEMS

The first series of problems deals with the state description of systems, as introduced in Section 5.2. Realizations are discussed in Section 5.3.

5.9.1. State description of a multiple delay. Consider the discrete-time multiple delay system with IO map

$$y(n) = u(n - M), \quad n \in \mathbb{Z},$$

with M a positive integer.

(a) Make it plausible that to determine the system behavior from time n on given the input $u(k)$ for $k \geq n$ it is necessary to know the M past values $u(n - M)$, $u(n - M + 1), \cdots, u(n - 1)$ of the input.
(b) For this reason, choose the state $x(n)$ at time n as $x(n) = \text{col}\,(u(n - M), u(n - M + 1), \cdots, u(n - 1))$, and determine the state difference and output equations of the system.
(c) Is the system linear? Is it time-invariant? If the answer is yes to both questions, then represent the system in the standard form $x(n + 1) = Ax(n) + Bu(n)$, $y(n) = Cx(n) + Du(n)$, $n \in \mathbb{Z}$.
(d) Set up a block diagram for the realization of the system using unit delays, adders, and gains.

5.9.2. Fibonacci equation as a state system. Consider a system described by the Fibonacci equation

$$y(n + 2) = y(n) + y(n + 1), \quad n \in \mathbb{Z}_+.$$

(a) Show that the state at time n may be chosen as $x(n) = \text{col}\,(y(n - 1), y(n))$.
(b) Derive the state difference and output equations of the system.
(c) Is the system linear? Is it time-invariant? If the answer to both questions is yes, represent the system in the standard form $x(n + 1) = Ax(n) + Bu(n)$, $y(n) = Cx(n) + Du(n)$, $n \in \mathbb{Z}_+$.
(d) Determine the block diagram for the realization of the system by using unit delays, adders, and gains.

5.9.3. State of a binary shift register. The binary shift register of Fig. 5.22 consists of a sequence of N binary memory elements that are connected in series as indicated. The input u consists of a sequence of bits $u(n)$, $n \in \mathbb{Z}_+$, that one by one shift into the left-most memory element. Each time a bit is entered, the entire contents of the register move one position to the right. The final bit moves out and is the output at that time.

(a) What is the state of this system?
(b) Determine the state difference and output equations.
(c) Is the system time-invariant?

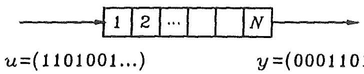


Figure 5.22. A binary shift register.

5.9.4. State description of a nonlinear RC network. The electrical network of Fig. 5.23 consists of a voltage source, which produces the input u to the system, two resistors, a capacitor, and an ideal diode. The output is the voltage across the capacitor.

(a) Make it plausible by an argument involving initial conditions that the state of the system is the voltage across the capacitor.
(b) Determine the state differential and output equations of the network.
(c) Show that the system is time-invariant but not linear.
(d) Give a block diagram for the realization of the system using integrators, adders, and function generators.

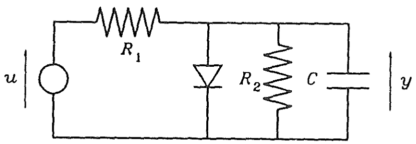


Figure 5.23. An RC network with a diode.

5.9.5. State representation of the van der Pol equation. The *van der Pol equation* is the nonlinear differential equation

$$y'' + \mu(y^2 - 1)y' + y = 0,$$

with μ a nonnegative constant. For $\mu = 0$ the equation reduces to what is often called the *harmonic oscillator*. If μ is nonzero, then the second term of the equation introduces damping, which is negative for $|y| < 1$ and positive if $|y| > 1$. The negative damping for small y keeps the circuit oscillating, while the positive damping for $|y| > 1$ more or less stabilizes the amplitude of the oscillation.

(a) Show that if $\mu = 0$ the basis solutions of the resulting linear differential equation are harmonics.
(b) Make it plausible by an argument involving initial conditions that the state at time t consists of $y(t)$ and its derivative $y'(t)$.
(c) Derive the corresponding state differential and output equations.
(d) Is the system linear? Is it time-invariant?
(e) Give a block diagram for the realization of the system by using integrators, adders, gains, and multipliers.

> Balthasar van der Pol was a Dutch physicist. He discovered his equation in the study of nonlinear effects in oscillator circuits with electronic vacuum tubes and described it in a classic paper in 1926.

5.9.6. State description of the double spring-mass system. The double spring-mass system of 4.8.3(b) (Fig. 4.18) comprises two masses and two springs. The output of system is now assumed to consist of the positions z_1 and z_2 of the two masses.

(a) The state variables of mechanical systems typically are *positions* and *velociti*... Therefore, choose the state of the system as $x = \text{col}(z_1, z_1', z_2, z_2')$, and dete... mine the state differential and output equations.
(b) Is the system linear? Is is time-invariant? If the answer is affirmative both questions, represent the system in the standard form $\dot{x} = Ax + B$... $y = Cx + Du$.
(c) Give a block diagram for the realization of the system using integrators, adder and gains.

5.9.7. Double integrator. The block diagram of Fig. 5.24 shows a *double integrator*, consisting of a series connection of two integrators. Determine a state representation of this system, including the state differential and output equations. Find the state transition map of the system.

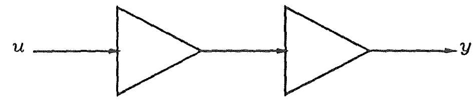


Figure 5.24. A double integrator.

5.9.8. Series and parallel connection of two state systems. Consider two continuous-time systems, whose state differential and output equations are of the form

$$\dot{x}_1(t) = A_1x_1(t) + B_1u_1(t),$$
$$y_1(t) = C_1x_1(t) + D_1u_1(t), \qquad t \in \mathbb{R},$$

and

$$\dot{x}_2(t) = A_2x_2(t) + B_2u_2(t),$$
$$y_2(t) = C_2x_2(t) + D_2u_2(t), \qquad t \in \mathbb{R},$$

respectively.

(a) Suppose that the systems are connected in series as in Fig. 5.25 (a), so that $u_2 = y_1$. For this to make sense we need u_2 and y_1 to have the same dimensions. Prove that the series connection is described by the combined state differential and output equations

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}A_1 & 0\\ B_2C_1 & A_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}B_1\\ B_2D_1\end{bmatrix}u(t),$$

$$y_2(t) = [D_2C_1 \quad C_2]\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + D_2D_1u(t), \qquad t \in \mathbb{R}.$$

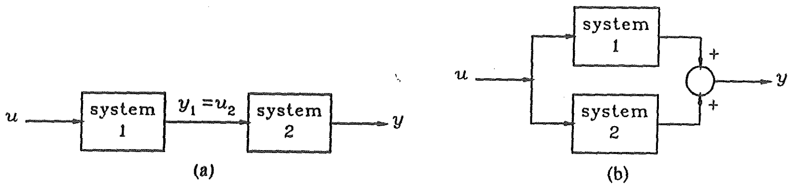


Figure 5.25. (a) series connection of two systems. (b) parallel connection.

(b) Find the combined state differential and output equations for the parallel connection of Fig. 5.25 (b). For this to make sense we need on the one hand u_1 and u_2 to have the same dimensions, and on the other y_1 and y_2 to have equal dimensions.

The state representation of difference and differential systems is treated in Section 5.3.

5.9.9. State representation of difference and differential systems.

(a) *Difference system without input differences.* Consider the difference system

$$y(n) + q_{N-1}y(n-1) + \cdots + q_0y(n-N) = p_0u(n), \qquad n \in \mathbb{Z},$$

with $q_0, q_1, \cdots, q_{N-1}$ and p_0 constant coefficients. Show that the state of the system at time t may be chosen as $x(n) = \text{col}\,(y(n-1), y(n-2), \cdots, y(n-N))$, and find the state difference and output equations.

(b) *Differential system without input derivatives.* Consider the differential system

$$y^{(N)}(t) + q_{N-1}y^{(N-1)}(t) + \cdots + q_0y(t) = p_0u(t), \qquad t \in \mathbb{R},$$

with $q_0, q_1, \cdots, q_{N-1}$ and p_0 constant coefficients. Show that the state of the system at time t may be chosen as $x(t) = \text{col}\,(y(t), y^{(1)}(t), \cdots, y^{(N-1)}(t))$, and find the state differential equation and output equations.

(c) *Backward differencer.* Determine a state representation of the backward differencer of Problem 3.10.5, described by

$$y(n) = u(n) - u(n-1), \qquad n \in \mathbb{Z}.$$

(d) Differential system. Find a state representation of the differential system of Problem 4.8.6(h), which is given by

$$y''(t) - y(t) = u'(t) - u(t), \qquad t \in \mathbb{R}.$$

The solution of state difference and differential equations is the subject of Section 5.4.

5.9.10. Existence of the solution of a state differential equation. Consider the system with state differential equation

$$\dot{x}(t) = \sqrt{|x(t)|}, \qquad t \in \mathbb{R},$$

with the initial condition $x(0) = x_0 \neq 0$.

(a) Use 5.4.1 to show that the differential equation has a unique solution on some interval $[-\eta, \eta]$ with $\eta > 0$.

(b) Use separation of variables to find this solution.

The solution of linear state difference and differential equations is extensively discussed in Section 5.5.

5.9.11. State transition matrix of the harmonic oscillator. The harmonic oscillator (see Problem 5.9.5) is described by the state differential equation

$$\dot{x}(t) = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}x(t), \qquad t \in \mathbb{R}.$$

Determine the state transition matrix Φ of the system by summing the infinite sum for e^{At}.

5.9.12. Solution of the state equations of the discrete-time multiple delay system. The state equations of the discrete-time delay system of Problem 5.9.1 are of the form $x(n+1) = Ax(n) + Bu(n)$, $y(n) = Cx(n) + Du(n)$, $n \in \mathbb{Z}$, where A is an $M \times M$ matrix, B an $M \times 1$ matrix, C a $1 \times M$ matrix and D a 1×1 matrix as follows

$$A = \begin{bmatrix}0 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & 0 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}; \qquad B = \begin{bmatrix}0\\ 0\\ \cdots\\ 0\\ 1\end{bmatrix},$$

$$C = [1 \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0], \qquad D = 0.$$

(a) Find the state transition matrix Φ of the system.

(b) Given the state transition matrix, find the impulse response of the system. Is the result surprising?

In Section 5.6 it is shown that modal analysis, or diagonalization of the matrix A, may simplify the analysis of time-invariant state difference and differential systems.

5.9.13. Modal analysis of the Fibonacci system. The state representation of the Fibonacci system of Problem 5.9.2 is of the form $x(n+1) = Ax(n)$, $n \in \mathbb{Z}_+$, with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix A.
(b) Determine the modes of the system.
(c) For each mode, find initial conditions such that the mode is excited without exciting the other modes.
(d) Compute the transition matrix Φ of the system using the modal transformation.

5.9.14. Modal analysis of the harmonic oscillator. The harmonic oscillator (see Problem 5.9.11) is described by the state differential equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t), \qquad t \in \mathbb{R}.$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix A.
(b) Determine the modes of the system, in real form.
(c) Compute the transition matrix Φ of the system by using the modal transformation.

5.9.15. Modal analysis of the double spring-mass system. The state differential and output equations of the double spring-mass system of Problem 5.9.6 are of the form $\dot{x} = Ax + Bu$, $y = Cx + Du$, where A is a 4×4 matrix, B a 4×1 matrix, C a 1×4 matrix, and D a 1×1 matrix as follows

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\omega_1^2 + \dfrac{m_2}{m_1}\omega_2^2\right) & 0 & \dfrac{m_2}{m_1}\omega_2^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_2^2 & 0 & -\omega_2^2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \omega_1^2 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad D = 0,$$

where $\omega_1^2 = k_1/m_1$ and $\omega_2^2 = k_2/m_2$. Assume the numerical values $\omega_1^2 = 3/2$, $\omega_2^2 = 4/3$ and $m_2/m_1 = 1/8$.

(a) Find the eigenvalues and corresponding eigenvectors of the matrix A.
(b) Determine the modes of the system, in real form.
(c) For each mode, find initial conditions such that the mode is excited without exciting the other modes. Which physical motion of the system corresponds to each of the modes?
(d) Compute the transition matrix Φ of the system by using the modal transformation.
(e) Given the state transition matrix, determine the impulse response matrix of the system.

5.9.16. RCL network. Use the modal transformation to compute the transition matrix of the fourth-order RCL system of Example 5.6.12(b) for the given numerical values.

BIBO, CICO, BIBS, and CICS stability of state systems are discussed in Section 5.7.

5.9.17. BIBO, CICO, BIBS, and CICS stability. Determine whether the following st systems are BIBO, CICO, BIBS, and CICS stable.

(a) The harmonic oscillator of Problem 5.9.11.
(b) The double spring-mass system of Problem 5.9.15.
(c) The multiple delay system of Problem 5.9.12.
(d) The nonlinear system with state differential and output equations

$$\dot{x}(t) = \alpha[u(t) - x^2(t)],$$
$$y(t) = x(t), \qquad t \in \mathbb{R},$$

with α a positive constant. This is the moving car as represented in Proble 4.8.1.

(e) The state differential system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, with

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad C = [1 \quad -1].$$

5.9.18. Stability of the parallel and series connections of two linear time-invariant stat differential systems.

(a) Prove that the parallel connection of two linear time-invariant state differential systems, as discussed in Problem 5.9.8(b), is BIBS stable if and only if each of the two subsystems is BIBS stable. Likewise, prove that the parallel connection is CICS stable if and only each of the component systems is CICS stable.
(b) What can be said about the BIBS and CICS stability of the *series* connection in terms of the BIBS or CICS stability of the component systems?

The frequency response of linear time-invariant state difference and differential systems, finally, is treated in Section 5.8.

5.9.19. Frequency response.

(a) Consider the multiple delay system as represented in Problem 5.9.12, and let $M = 3$. Verify that the sufficient conditions for the existence of the frequency response function are satisfied, and determine the frequency response function. Is the result surprising? What is the frequency response function for arbitrary M?
(b) Show that the system of Problem 5.9.17(e) has a well-defined steady-state response to any harmonic input $u(t) = e^{j2\pi ft}$, $t \geq 0$, of the form $y(t) = \hat{h}(f)e^{j2\pi ft}$, $t \geq 0$. Determine the frequency response function $\hat{h}$.

5.9.20. Electrical circuit. The electrical network of Fig. 5.26 contains one inductor and two capacitors. Its input is the voltage u of the voltage source, and its output y is the voltage across the capacitor C_2.

(a) Take the charges of the capacitors and the flux contained by the inductor as components of the state, and derive the state differential and output equations of the network. (Alternatively, take the voltages across the capacitors and the current through the inductor as components of the state.)

(b) Determine the frequency response function of the network.

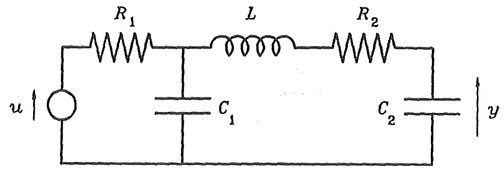


Figure 5.26. An electrical network.

5.10 COMPUTER EXERCISES

The computer exercises for this chapter involve the simulation of various discrete- and continuous-time state systems.

5.10.1. Discrete-time delay and moving averager. Consider the discrete-time delay, described by

$$y(n) = u(n - N), \qquad n \in \mathbb{Z},$$

and the simple moving averager

$$y(n) = u(n) + u(n - 1) + \cdots + u(n - N), \qquad n \in \mathbb{Z}.$$

(a) Find the state difference equations describing these two systems.
(b) Let $N = 3$ and assume that the systems are at rest at time 0. Use the state difference equations to compute and plot the response of these two systems on the time axis $\{0, 1, \cdots, 100\}$ to a pure noise input. *Hint:* In SIGSYS, generate the input by the command u=noiseplus after having defined the appropriate time axis.

5.10.2. Hénon's equation. The solutions of certain nonlinear (state) difference equations (without input) exhibit a highly irregular behavior, which is said to be "chaotic." A well-known example is *Hénon's equation,* which is given by

$$\begin{aligned} x_1(n + 1) &= 1 - ax_1(n)^2 + x_2(n), \\ x_2(n + 1) &= bx_1(n), \qquad n \in \mathbb{Z}_+, \end{aligned}$$

with $a = 1.4$ and $b = 0.3$. (Chaotic behavior is not obtained for all values of a and b.)

(a) Solve the difference equations numerically for $n \in \{0, 1, \cdots, N\}$, with $N = 100$ or 200. Observe that if the initial conditions $x_1(0)$ and $x_2(0)$ are chosen too large (one of the two or both in absolute value greater than 1 or 2), then the solution is unstable, but that for small initial conditions chaotic behavior is obtained. Plot the solutions as a function of time.
(b) Next plot the solution in the (x_1, x_2)-plane rather than as a function of time. Observe that the points form an interesting pattern, which is not at all discernible in the chaotic behavior of the time signals. The pattern is clearer when the (x_1, x_2) pairs are plotted individually, without connecting lines. *Hint:* In SIGSYS, a real signal x_2 may be plotted against a real signal x_1 by the command polarplot x1+j*x2.

> *Chaos* is a popular subject among physicists and mathematicians, because it is thought to be a model for important physical phenomena such as turbulence. Hénon's equation was published in 1976. (M. Hénon, "A Two-Dimensional Mapping with a Strange Attractor." *Comm. Math. Phys.*, Vol. **50**, 69–77, 1976.)

5.10.3. Lotka-Volterra prey-predator model. The *Lotka-Volterra* equations model an ecological system consisting of two populations that inhabit an isolated environment. The first population, the *preys,* live on a vegetarian diet. The other population, the *predators,* feed entirely on the preys. The growths or declines of the populations are mutually dependent. Let x denote the size of the predator population, and y that of the prey population. Then the Lotka-Volterra model states that x and y change with time according to the equations

$$\begin{aligned} \dot{x} &= -ax + bxy, \\ \dot{y} &= cy - dxy, \end{aligned} \tag{1}$$

with a, b, c, and d constants. The quantity $\dot{x}$ is the rate of increase of the predator population. The term $-ax$ on the right-hand side of the first equation indicates the natural decline of the predator population in the absence of prey. The term bxy is the growth of the predator population, which is proportional both to the size of the predator population itself and to that of the prey population. The term cy on the right-hand side of the second equation is the natural growth of the prey population when left to itself, while the term $-dxy$ represents its decline due to the predators. The equations (1) form the state differential equation for the model.

> Lotka and Volterra independently proposed their population model in the 1920s. Vito Volterra (1860–1940) was an Italian mathematican who worked on differential and integral equations. The work of the actuarian and demographer Alfred J. Lotka extends from the first decennium of this century until the 1940s.

(a) Show that the system has two equilibrium states in the (x, y)-plane, one at the origin and one in the first quadrant (assuming that the constants a, b, c, and d are all positive). The following will show that the equilibrium state at the origin is unstable (meaning that all solutions starting close to it tend to move away), while the other equilibrium state is stable (i.e., solutions starting close to it stay close).
(b) Assume $a = b = c = d = 1$, and determine the stable equilibrium state. Solve the Lotka-Volterra equations numerically on the time interval [0, 10] for the following initial conditions:

(b.1) $x(0) = y(0) = 2,$

(b.2) $x(0) = y(0) = 1.1,$

(b.3) $x(0) = y(0) = 0.1.$

Plot the population sizes x and y both against time and against each other. The populations exhibit a periodic behavior. Explain this behavior qualitatively. *Hint:* To plot a real signal y versus a real signal x, in SIGSYS the command polarplot x+j*y may be used.

(c) Ascertain the effect of changing the parameter c from 1 to 2 on the equilibrium state and the trajectories. The change means that the natural growth of the prey population is twice as large as before. Explain the results of this change qualitatively.

> **Numerical solution of differential equations.**
> Refer to the box with this title in Section 4.9 for recommendations on the numerical integration of differential equations.

5.10.4. Van der Pol's equation. (Compare 5.9.5.) Van der Pol's equation

$$y'' + \mu(y^2 - 1)y' + y^2 = 0,$$

is a model for a nonlinear oscillator. The nonlinearity causes the oscillator to have negative damping for small values of y, and an increasingly large positive damping if y exceeds the value 1. The effect of the nonlinearity depends on the magnitude of the constant μ. Defining $x_1 = y$ and $x_2 = y'$ the second-order equation may be rewritten in the form of the state differential equation

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -\mu(x_1^2 - 1)x_2 - x_1.$$

(a) For $\mu = 0$ the system reduces to a harmonic oscillator. Determine the period P of the harmonic.

(b) Integrate the differential equations for $\mu = 0$ on the interval $[0, a]$ with the initial conditions $y(0) = 2.1$ and $y'(0) = 0$. Here a is $2P$ rounded up to a "nice" value. Plot $x_1 = y$ and $x_2 = y'$ both against time and against each other. *Hint:* To plot a real signal x_2 versus a real signal x_1, in SIGSYS the command polarplot x1+j*x2 may be used.

(c) Repeat (b) for $\mu = 0.1$, 1 and 10. Use the same initial conditions, and increase the length of the integration interval as much as needed to allow the behavior of the solution to become stationary. The larger μ becomes, the "stiffer" the differential equation is (see box), so that it may be necessary to reduce the step size of the integration.

(d) For $\mu = 1$, study the effect of changing the initial conditions. Observe that in the (x_1, x_2)-plane whatever the initial conditions are the solution eventually keeps moving on a closed trajectory, which is called the *limit cycle* of the system. In this particular case the initial condition (2, 0) is always on the limit cycle.

5.10.5. Simulation of a simple demodulation circuit. A well-known technique in communication is *amplitude modulation*. Modulation is needed to convert low-frequency signals to high-frequency signals, which may be transmitted as radio waves or along a transmission line. The idea of amplitude modulation is the following. Suppose that the continuous-time message signal m is to be transmitted, and let the *carrier* c be the real harmonic signal given by

$$c(t) = \cos(2\pi f_c t), \qquad t \in \mathbb{R},$$

> **"Stiff" differential equations.**
> The numerical solution of the differential equation describing the demodulator of Exercise 5.10.5 is made difficult by the presence of the time constant R_1C. For the demodulator to function adequately, this time constant should be small, necessitating a small step size for the numerical integration. Differential equations containing time constants that widely differ in magnitude are said to be *stiff*. This name derives from mechanical systems, where stiff springs result in small time constants. Stiffness and nonlinearities are characteristic for digital electronic networks. The simulation of such networks, especially large scale networks, is numerically very demanding, and special software exists for this purpose.

with the carrier frequency f_c large. Then amplitude modulation of the carrier c with the message signal m results in the modulated signal u given by

$$u(t) = [m_o + m(t)] \cos(2\pi f_c t), \qquad t \in \mathbb{R}.$$

The number m_o is a positive constant such that $m_o + m(t) \geq 0$ for all t.

After the modulated signal u has been transmitted and received, *demodulation* is required to recover the message signal m. A common approximate demodulation scheme that is used in simple AM receivers is implemented by the network of Fig. 5.27. In Fig. 5.28 the output y of the network corresponding to a modulated input signal u is plotted.

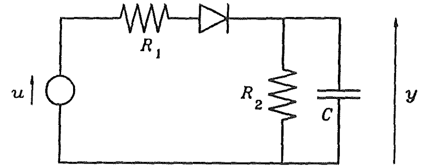


Figure 5.27. A simple demodulator circuit.

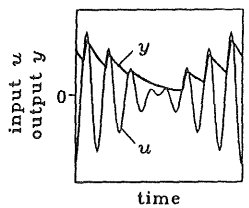


Figure 5.28. Input u and output y of the demodulator circuit.

The demodulated output exhibits a high-frequency ripple owing to imperfect removal of the carrier. The ripple may be reduced by choosing the carrier frequency large. Choosing the carrier frequency large also allows to make the time constant R_2C small which results in a closer approximation of the original waveform.

(a) Describe the operation of the demodulation network qualitatively.
(b) Generate the message signal m as

$$m(t) = \begin{cases} 1 & \text{for } 0.25 \le t < 0.5 \text{ and } 0.75 \le t < 1, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in [0, 2).$$

Take the carrier frequency as $f_c = 25$ and generate the modulated signal u with $m_o = 0.25$. Plot u. *Hint:* Generate the continuous-time signals on a discrete time axis with sampling interval 0.0025 or less.
(c) Show that the demodulation network is described by the differential equation

$$\dot{y} = -\frac{1}{R_2C}y + \frac{1}{R_1C}\,\text{ramp}\,(u - y).$$

(d) Compute the response of the network to the input u by integrating the differential equation with the initial condition $y(0) = 0$ for $R_1C = 0.002$ and $R_2C = 0.05$. Plot the input u and the output y. Compute the response if R_2C is changed to 0.2 and discuss the difference. *Hints:* Consult the Tutorial to see how in SIGSYS time t need be integrated along with y to include the external input u in the differential equation. A Runge-Kutta integration scheme of order two with step size equal to the sampling interval yields adequate results.

5.10.6. Cruise control system. In 4.8.1 it was found that the moving car of Example 3.2.13 is described by the differential equation

$$\frac{dw(t)}{dt} = \alpha[u(t) - w^2(t)], \qquad t \ge 0, \tag{2}$$

where w is the normalized speed of the car, u the throttle position, and α a physical constant. Assume that the car is equipped with a cruise control system as in Fig. 1.13, which adjusts the throttle position according to

$$\frac{du(t)}{dt} = k[w_r(t) - w(t)], \qquad t \ge 0, \tag{3}$$

where w_r is the reference speed (again as fraction of the top speed), and k the "gain" of the cruise controller. The function of the cruise controller is to keep increasing the throttle position as long as the reference speed exceeds the actual speed, and to decrease the throttle position in the other case. Together, (2) and (3) form the state differential equation for the controlled system, with the reference speed w_r as external input.

The purpose of this exercise is to see whether this scheme works, and what the best value for the gain k is. In the following, take $\alpha = 1/10$.

(a) First choose $k = 0$ (i.e., the controller is inactive). Solve the state differential equation numerically on the interval [0, 100] with the initial conditions $w(0) = 0.5$ and $u(0) = 0.36$. For $k = 0$ the throttle position keeps the constant value $u_o = 0.36$. Observe how the car speed reaches the steady-state constant value $w_\infty = \sqrt{u_o} = \sqrt{0.36} = 0.6$.
(b) Determine the constant throttle position u_o that corresponds to the constant cruising speed $w_o = 0.5$. Solve the differential equations numerically with $w_r(t) = 0.6$ for $t \ge 0$ and the initial conditions $w(0) = w_o = 0.5$ and $u(0) = u_o$ for $k = 0.02$, 0.05, and 0.1. Compare the responses of the system to that of (a). What is the best value of k?

6

Expansion Theory and Fourier Series

6.1 INTRODUCTION

The next three chapters are devoted to *frequency domain* methods for the analysis of linear time-invariant systems. The methods for obtaining the response of a linear time-invariant system we studied so far rely on *convolutions*. Because the summation or integration implicit in convolution is over time, these methods constitute what is called *time domain analysis*.

Convolution suffers from the drawback that it is a complicated operation, which makes it difficult to assess the system behavior. Also, performing (discrete-time) convolution on a digital computer involves many multiplications and additions, which means that it is computationally expensive.

Both difficulties may to a large extent be avoided when using *frequency domain* analysis. Frequency domain analysis is based on the following two facts: First, surprisingly, practically every time signal may be written as a linear combination of (complex) harmonic signals with different frequencies. Second, the response of a linear time-invariant system to a complex harmonic input signal is the same harmonic multiplied by a gain, which is the value of the *frequency response function* of the system at that frequency.

Frequency domain methods exploit these two facts as follows. To determine the response of a linear time-invariant system we first *decompose* the input as a lin-

ear combination of harmonics with different frequencies. Next, the responses to t individual harmonics are obtained, which again are harmonic. Finally, by the supe position property of linear systems, the harmonic responses are linearly combined obtain the total response.

Frequency domain analysis has considerable advantages. It has intuitive ap peal, and moreover leads to efficient computer algorithms, notably the *fast Fouri transform* (FFT), for signal processing.

We start this chapter by developing in Section 6.2 the theory of *signal expan sion* in an abstract setting. The idea is to decompose a given signal as a linear com bination of *basis signals*. In particular, *orthogonal* expansions are emphasized. In Section 6.3 it is shown that for the analysis of the response of a linear system the *spectral basis* is the most suitable basis. For linear time-invariant systems the spec tral basis turns out to consist of harmonics.

In Section 6.4 we develop the expansion of *periodic* signals in harmonics. This leads to the finite and infinite *Fourier series* expansions. Section 6.5 describes how the Fourier series expansion may be used to analyze the response of discrete- and continuous-time linear time-invariant systems to periodic inputs.

In Chapter 7 the frequency domain approach is extended to *aperiodic* signals based on *Fourier integral* theory. The *Laplace* and *z-transforms*, introduced in Chapter 8, are further developments that allow the application of frequency domain methods to constant coefficient linear difference and differential systems.

6.2 SIGNAL EXPANSION

In this section we explore how signals may be represented as linear combinations of a number of fixed signals, together called a *basis*. Special attention is given to expansions in *orthogonal* bases.

Linear Independence

We first define linear *dependence* and *independence*.

6.2.1. Definition: Linear dependence and independence. Suppose that X is a linear space over the field $\mathcal{F}$ of real or complex scalars, and that S is a subset of X.

(a) The element $y \in X$ is a *finite linear combination* of the elements $x_1, x_2, \cdots, x_N$ of X if there exist scalars $\alpha_1, \alpha_2, \cdots, \alpha_N \in \mathcal{F}$ such that

$$y = \sum_{i=1}^{N} \alpha_i x_i.$$

(b) The element $y \in X$ is *linearly dependent* on S if y can be expressed as a finite linear combination of elements in S. Otherwise, y is said to be *linearly independent* of S.

11.5.3. Nonlinear feedback system. In the feedback system of Fig. 11.39, the block marked "H" is a linear time-invariant system with transfer function

$$H(s) = \frac{1}{s(s+1)},$$

while the block marked "f" is memoryless nonlinear with IO map

$$f(e) = k \text{ sign } (e),$$

with k a positive constant gain.

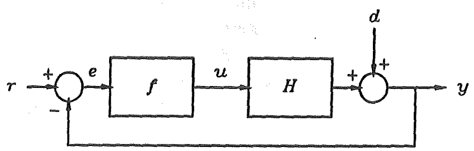


Figure 11.39. A nonlinear feedback system.

(a) Simulate the system with zero external input r and zero disturbance d for various initial conditions and different values of k. Does it look as if the closed-loop system is stable for any positive gain k? *Hints:* Determine a state representation for the linear system. Simulate over the interval [0, 10]. With Runge-Kutta 2 integration a step size of 0.1 seems to give adequate results.

(b) Let $k = 1$. Determine the response of the system to a unit step in the external input r and in the disturbance d. Comment on the results. Does it help to increase the gain k to 2?

11.5.4. Control system design. Consider the linear time-invariant feedback system of Fig. 11.40, where the plant has the transfer function

$$H(s) = \frac{1}{(1+sT_1)(1+sT_2)},$$

with $T_1 = 1$ [s] and $T_2 = 0.1$ [s]. We consider two possibilities for the transfer function G of the forward compensator: a *pure gain*

$$G(s) = k,$$

and a so-called *lag-lead filter*

$$G(s) = k\frac{1+sT_3}{1+sT_4}.$$

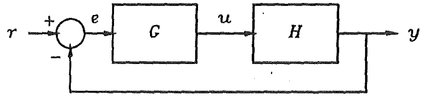


Figure 11.40. A linear time-invariant feedback system.

(a) First consider the pure gain compensator. Show by inspection of the Nyquist plot of the feedback system for $k = 1$ that the closed-loop system is stable for all positive k. Also show that as k increases the gain margin remains constant but the phase margin becomes smaller and smaller. The closed-loop transfer function of the system is given by $H_{cl} = L/(1+L)$, with $L = HG$. Compute and plot the closed-loop frequency response $\hat{h}_{cl}(f) = H_{cl}(j2\pi f), f \in \mathbb{R}$, for several values of the gain k. Select a value of k for which the bandwidth of the closed-loop system is as large as possible without undesirable peaking of the closed-loop frequency response function. Compute the corresponding impulse response of the closed-loop system by inverse Fourier transformation. From this, compute the step response of the closed-loop system, and plot it.

(b) Repeat all this for the lag-lead compensator, with $T_3 = 1$ [s] and $T_4 = 0.1$ [s]. Show that by selecting a suitable gain k the lag-lead compensator results in a faster closed-loop response.

Supplement A

A Review of Complex Numbers, Sets, and Maps

Complex numbers play an important role in the theory of signals and systems. We therefore summarize in the first part of this supplement the most important properties of complex numbers and their operations. It also turns out to be very useful to employ some simple notions and notations connected with sets and maps. These are reviewed in the remainder of the supplement.

Complex Numbers

A *complex number* z is a pair of real numbers x and y written as

$$z = x + jy.$$

The real number x is called the *real part*, and the real number y the *imaginary part* of z. This is expressed by the notations

$$x = \text{Re}\,(z), \qquad y = \text{Im}\,(z).$$

If $y = 0$ we usually write $z = x$; if $x = 0$ we write $z = jy$.

Complex numbers have special rules for addition and multiplication. If

$$z_1 = x_1 + jy_1 \qquad \text{and} \qquad z_2 = x_2 + jy_2$$

are two complex numbers, their sum and product are defined as

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2),$$
$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1).$$

It follows from the multiplication rule that

$$j^2 = -1.$$

The complex numbers, like the real numbers, constitute what is called a *field*. A fie is a set with two operations, in this case addition and multiplication, satisfying number of hypotheses. These include several commutativity, associativity, and di tributivity properties, the existence of a *zero element* and a *unit element*, and that c the *negative* and the *reciprocal* of nonzero elements. The zero element of the com plex field is $0 + j0 = 0$, while the unit is $1 + j0 = 1$. The negative o $z = x + jy$ is

$$-z = -x + j(-y),$$

while if $z \neq 0$ its reciprocal is

$$z^{-1} = \frac{x - jy}{x^2 + y^2}.$$

Two complex numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ may be subtracted as

$$\begin{aligned} z_1 - z_2 &= z_1 + (-z_2) \\ &= (x_1 - x_2) + j(y_1 - y_2), \end{aligned}$$

and if $z_2 \neq 0$ they may be divided as

$$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\frac{-x_1y_2 + x_2y_1}{x_2^2 + y_2^2},$$

although division is more easily done using the polar representation that is discussed next.

A complex number $z = x + jy$ may be represented by a point in a plane with Cartesian coordinates (x, y) as in Fig. A.1. The plane is called the *complex plane, and*

$$z = x + jy$$

the *Cartesian* representation of the complex number. An alternative way of representing the complex number is in terms of its *polar* coordinates, formed by its *mag-*

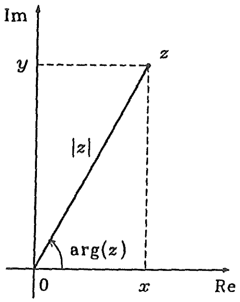


Figure A.1. Complex number as a point in a plane.

nitude $|z|$, and its *argument* arg (z), as indicated in Fig. A.1. The magnitude is also referred to as the *modulus* or *absolute value* of the complex number. The argument is sometimes called the *angle* or *phase* of the complex number. The *polar representation of* z is

$$z = |z| e^{j \arg(z)}.$$

The relations between the real and imaginary parts x and y and the magnitude and argument of $z = x + jy$ are

$$|z| = \sqrt{x^2 + y^2}, \qquad \arg(z) = \begin{cases} \operatorname{atan}\left(\dfrac{y}{x}\right) & \text{if } x > 0, \\ \operatorname{atan}\left(\dfrac{y}{x}\right) + \pi & \text{if } x \leq 0 \text{ and } y > 0, \\ \operatorname{atan}\left(\dfrac{y}{x}\right) - \pi & \text{if } x \leq 0 \text{ and } y \leq 0, \\ \text{undetermined} & \text{if } x = 0 \text{ and } y = 0, \end{cases}$$

$$x = |z| \cos(\arg(z)), \qquad y = |z| \sin(\arg(z)),$$

with atan denoting the arctangent.

Addition and subtraction of complex numbers are most easily performed using the Cartesian representation, while multiplication and division are simplest when the polar forms are available, because

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| e^{j(\arg(z_1) + \arg(z_2))},$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{j(\arg(z_1) - \arg(z_2))}.$$

The adjective Cartesian derives from the name of the French philosopher, mathematician, and natural scientist René Descartes (1596–1650).

The *complex conjugate* $\overline{z}$ of a complex number $z = x + jy$ is the complex number

$$\overline{z} = x - jy.$$

It is easily verified that

$$z \cdot \overline{z} = |z|^2.$$

Finally, any two complex numbers z_1 and z_2 satisfy the *triangle inequality*

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

A.1. Example: Operations with complex numbers. Let z_1 and z_2 be the complex numbers

$$z_1 = 1 + j, \qquad z_2 = 1 + j\sqrt{3},$$

so that

$$z_1 + z_2 = 2 + j(1 + \sqrt{3}), \qquad z_1 - z_2 = j(1 - \sqrt{3}).$$

We have

$$|z_1| = \sqrt{1 + 1} = \sqrt{2}, \qquad |z_2| = \sqrt{1 + 3} = 2,$$
$$\arg(z_1) = \operatorname{atan}(1) = \pi/4, \qquad \arg(z_2) = \operatorname{atan}(\sqrt{3}) = \pi/3,$$

so that the polar representations of z_1 and z_2 are

$$z_1 = \sqrt{2} e^{j\pi/4}, \qquad z_2 = 2e^{j\pi/3}.$$

From the polar representation we find easily

$$z_1 \cdot z_2 = 2\sqrt{2} e^{j7\pi/12}, \qquad \frac{z_1}{z_2} = \frac{1}{2}\sqrt{2} e^{-j\pi/12}.$$

A.2. Exercise. Complex exponential and logarithm.

(a) If $z = x + jy$ is a complex number, the complex exponential is given by

$$e^z = e^x(\cos(y) + j \sin(y)).$$

Prove

$$|e^z| = e^x, \qquad \arg(e^z) = y,$$

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

(b) If $z \neq 0$ is a complex number, the complex (natural) logarithm of z is

$$\log(z) = \log(|z|) + j\arg(z).$$

Prove that if $z_1z_2 \neq 0$

$$\log(z_1z_2) = \log(z_1) + \log(z_2),$$

and

$$\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2).$$ ■

Sets

We denote by

$$X = \{a, b, c, \cdots\}$$

the set X whose elements are $a, b, c, \cdots$. The familiar notations $x \in X$ and $x \notin X$ are assumed to be known, as well as the inclusion symbols $\subset$ and $\supset$ and the elementary set operations union $\cup$ and intersection $\cap$. We use the following notations for some well-known sets:

$\mathbb{N}$ the set of all natural numbers
$\mathbb{Z}$ the set of all integers,
$\mathbb{R}$ the set of all real numbers,
$\mathbb{C}$ the set of all complex numbers.

By $\{x \in X \mid P(x)\}$ we denote the subset of X that consists of elements $x \in X$ for which the proposition $P(x)$ holds.

A.3. Example: The set of nonnegative integers. The set

$$\mathbb{Z}_+ := \{x \in \mathbb{Z} \mid x \geq 0\}, \tag{1}$$

consists of all nonnegative integers. ■

The notation $a := b$ used in (1) indicates "assign the meaning of b to a", or "a equals b by definition."

The set $X_1 \times X_2 \times \cdots \times X_N$, called the *product set* of the sets $X_1, X_2, \cdots$ X_N, is the set of N tuples $(x_1, x_2, \cdots, x_N)$ with $x_1 \in X_1, x_2 \in X_2, \cdots, x_N \in$ The product set $X \times X \times \cdots \times X$, with X repeated N times, is written as X^N.

A.4. Example: The product sets $\mathbb{R}^N$ and $\mathbb{C}^N$. The sets $\mathbb{R}^N = \mathbb{R}$ $\mathbb{R} \times \cdots \times \mathbb{R}$, with $\mathbb{R}$ repeated N times, and $\mathbb{C}^N = \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}$, ag with $\mathbb{C}$ repeated N times, consist of all N tuples $(x_1, x_2, \cdots, x_N)$ with $x_1, x_2, \cdots$ x_N real or complex numbers, respectively. Sometimes it is useful to arrange the N ple $(x_1, x_2, \cdots, x_N)$ as a column vector

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{bmatrix}.$$

An equivalent but space saving notation for this column vector i $\operatorname{col}(x_1, x_2, \cdots, x_N)$. The set consisting of all such column vectors is also denote $\mathbb{R}^N$ or $\mathbb{C}^N$, depending on whether the elements are real or complex.

If a set has finitely many elements, then it is called a *finite* set. If it has countabl infinitely many elements, then the set is *countably infinite*. In all other cases it is *uncountable*.

A.5. Example: Bits. *Bits* are the elements of the finite set $\mathbb{B} := \{0, 1\}$. ■

A.6. Example: Bytes. Four-bit *bytes* are the elements of the finite product set $\mathbb{B}^4 = \{0000, 0001, 0010, \cdots, 1111\}$. ■

Typical examples of countably infinite sets are the set $\mathbb{N}$ of all natural numbers and the set $\mathbb{Z}$ of all integers. The best known examples of uncountable sets are the set of real numbers $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$.

A.7. Example: Bit streams. An infinite sequence of zeros and ones, such as $01001100\cdots$ is called a *bit stream*. The set of *all* bit streams is uncountable. To see this, note that any bit stream can be taken as the mantissa, in binary form, of a number between 0 and 1. As the numbers between 0 and 1 are uncountable, so are the bit streams. ■

Maps

A *map* ϕ from the set X to the set Y assigns to every element $x \in X$ a unique element $\phi(x) \in Y$. We write

$$\phi: X \to Y,$$

and call $\phi(x)$ the *image* of $x \in X$ under ϕ or the *value* of ϕ at x. A map is sometimes also referred to as a *function*, an *operator*, or a *transformation*, depending on the context. The set X is called the *domain* of the map ϕ. The set of all images of elements in X under ϕ is the *range* of the map.

A.8. Example: Real function of a real variable. Let ϕ: $\mathbb{R} \to \mathbb{R}$ be the function defined by $\phi(x) = x^2$. The function has domain $\mathbb{R}$ and range $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. ∎

If the range of the map ϕ is the whole set Y, then we say that ϕ is an *onto* or *surjective* map. If no two different elements of X have the same image, that is, if for $x_1 \in X$ and $x_2 \in X$

$$\phi(x_1) = \phi(x_2) \quad \text{implies} \quad x_1 = x_2,$$

then ϕ is said to be a *one-to-one* or *injective* map. If ϕ is both surjective and injective it is called *bijective*.

A.9. Examples: Injective, surjective, and bijective maps.

(a) The map ϕ: $\mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x^2$ is neither surjective (because its range is not the entire set $\mathbb{R}$) nor injective (because $-x$ and x map to the same number).

(b) As in 2.3.4, define the *entier* function int: $\mathbb{R} \to \mathbb{Z}$ such that int (x) is the largest integer N with the property that $N \leq x$. The map int: $\mathbb{R} \to \mathbb{Z}$ is surjective (because its range is the whole set $\mathbb{Z}$) but not injective (because every $x \in \mathbb{R}$ such that $N \leq x < N + 1$ has the same image N.)

(c) Finally, let ρ: $\mathbb{R} \to \mathbb{R}$ be defined by $\rho(x) = x^3$. As ρ is both injective and surjective, it is bijective. ∎

If ϕ and ψ are maps

$$\phi: X \to Y, \qquad \psi: U \to V,$$

with $V \subset X$, then the *composition* of ϕ and ψ is the map $\phi \circ \psi$: $U \to Y$ given by

$$(\phi \circ \psi)(u) = \phi(\psi(u)) \qquad \text{for all} \quad u \in U.$$

The operation $\circ$ is called *map composition*.

A.10. Example: Composition of maps. Let ϕ: $\mathbb{R} \to \mathbb{R}$ be the map defined by $\phi(x) = x^2$, and ψ: $\mathbb{R} \to \mathbb{Z}$ the map given by $\psi(x) = \text{int}\,(x)$. Then $\phi \circ \psi$: $\mathbb{R} \to \mathbb{R}$ is the composite map defined by $(\phi \circ \psi)(x) = [\text{int}\,(x)]^2$, while $\psi \circ \phi$: $\mathbb{R} \to \mathbb{Z}$ is the composite map given by $(\psi \circ \phi)(x) = \text{int}\,(x^2)$. Note that the maps $\phi \circ \psi$ and $\psi \circ \phi$ are *not* the same: the composition operation in general is not commutative. ∎

A bijective map ϕ: $X \to Y$ has an *inverse* map

$$\phi^{-1}: Y \to X,$$

that satisfies

$$(\phi^{-1} \circ \phi)(x) = x \qquad \text{for all} \quad x \in X,$$
$$(\phi \circ \phi^{-1})(y) = y \qquad \text{for all} \quad y \in Y.$$

A.11. Example: A map and its inverse. The maps defined in Examples A.9(a) and (b) have no inverses because they are not bijective. The map ρ: $\mathbb{R} \to \mathbb{R}$ given by $\rho(x) = x^3$ of A.9(c) is bijective. Its inverse ρ^{-1}: $\mathbb{R} \to \mathbb{R}$ is given by $\rho^{-1}(y) = y^{1/3}$. ∎

Power Sets

The set of all maps from the set X into the set Y is denoted by Y^X and is called a *power set*. This notation is effective in denoting sets of signals.

A.12. Examples: Power sets.

(a) The power set

$$\mathbb{C}^{\{0, 1, \cdots, N-1\}}$$

is the set of all maps

$$\{0, 1, \cdots, N - 1\} \to \mathbb{C}.$$

Thus, an element x of this power set is a map that assigns a complex number $x(i)$ to each $i \in \{0, 1, \cdots, N - 1\}$. Hence, specifying the map x is the same as specifying the Ntuple

$$(x(0), x(1), \cdots, x(N - 1)).$$

It follows that $\mathbb{C}^{\{0, 1, \cdots, N-1\}}$ is identical to the product set $\mathbb{C}^N$, which also consists of all Ntuples of complex numbers.

(b) Similarly, $\mathbb{C}^{\mathbb{Z}}$, the set of all maps x: $\mathbb{Z} \to \mathbb{C}$, may be identified with the set of all infinite sequences of complex numbers of the form $x = (\cdots, x(-1), x(0),$

$x(1), \cdots$), with $x(i) \in \mathbb{C}$ for each $i \in \mathbb{Z}$. As the set $\mathbb{C}^{\mathbb{Z}}$ appears frequently, we refer to it by the special notation

$$\ell = \mathbb{C}^{\mathbb{Z}}.$$

(c) Finally, the power set $\mathbb{C}^{\mathbb{R}}$ is the set of all complex-valued functions of a real variable. A typical element x of $\mathbb{C}^{\mathbb{R}}$ is a function $x(t)$, $t \in \mathbb{R}$, where $x(t) \in \mathbb{C}$ for each $t \in \mathbb{R}$. As also the set $\mathbb{C}^{\mathbb{R}}$ frequently makes its appearance, we denote it as

$$\mathscr{L} = \mathbb{C}^{\mathbb{R}}.$$

■

Supplement B

A Review of Linear Spaces, Norms and Inner Products

In this supplement we review the basic definitions of linear spaces, norms, and inner products.

Linear Spaces

Before giving the definition of a linear space we recall that the set of reals $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$, with the usual operations of addition and multiplication, both form a *field*. A field, roughly, is a set with two binary operations, called addition and multiplication, having all the usual properties such as commutativity, associativity and distributivity. Besides $\mathbb{R}$ and $\mathbb{C}$ also the set $\underline{N} = \{0, 1, \cdots, N-1\}$, with N a prime and addition and multiplication defined modulo N, is a field. Elements of a field are called *scalars*. The sum of two scalars α and β is denoted as $\alpha + \beta$, their product as $\alpha \cdot \beta$ or $\alpha\beta$. The unit of the field is written as 1 and its zero as 0.

Linear spaces are sets where the operations of *multiplication* of any element *by a scalar* and *addition* of any two elements are defined, satisfying a number of properties.

B.1. Definition: Linear space. Consider a field $\mathscr{F}$ and the triple $(X, +, \cdot)$, where X is a set, $+$ a binary operation

$$+ : X \times X \to X,$$

called *addition,* and $\cdot$ a binary operation

$$\cdot : \mathcal{F} \times X \to X,$$

called *multiplication by a scalar*. The image of $(x, y) \in X \times X$ under $+$ is denoted $x + y$, and the image of $(\alpha, x) \in \mathcal{F} \times X$ under $\cdot$ is denoted $\alpha \cdot x$ or αx. Then, $(X, +, \cdot)$ is a linear space over the field $\mathcal{F}$ if the following conditions are satisfied:

(a) Addition of two elements of X has the following properties:

(i) $(x + y) + z = x + (y + z)$ for all $x, y, z \in X$ (associativity),
(ii) $x + y = y + x$ for all $x, y \in X$ (commutativity),
(iii) there exists a zero element $\theta \in X$ so that $x + \theta = x$ for all $x \in X$,
(iv) for every $x \in X$ there exists an element $-x$ so that $x + (-x) = \theta$.

(b) Multiplication of an element of X by a scalar has the following properties:

(i) $\alpha(\beta x) = (\alpha\beta)x$ for all $\alpha, \beta \in \mathcal{F}$ and all $x \in X$ (associativity),
(ii) $1 \cdot x = x$ and $0 \cdot x = \theta$ for all $x \in X$,
(iii) $\alpha(x + y) = \alpha x + \alpha y$ for all $\alpha \in \mathcal{F}$ and all $x, y \in X$ (distributivity),
(iv) $(\alpha + \beta)x = \alpha x + \beta x$ for all $\alpha, \beta \in \mathcal{F}$ and all $x \in X$ (distributivity). ∎

Customarily the zero element θ is written as 0, not to be confused with the zero of the field $\mathcal{F}$. When it is clear what $+$, $\cdot$, and $\mathcal{F}$ are, usually the set X itself is referred to as a linear space. The elements of linear spaces are often called *vectors,* and linear spaces are sometimes referred to as *vector spaces.*

The sets $\mathbb{R}^N$ and $\mathbb{C}^N$ are the best known examples of linear spaces.

B.2. Example: The linear spaces $\mathbb{R}^N$ and $\mathbb{C}^N$. Given two elements $x = (x_1, x_2, \cdots, x_N)$ and $y = (y_1, y_2, \cdots, y_N)$ of $\mathbb{R}^N$ or $\mathbb{C}^N$, define addition in the usual way as $x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_N + y_N)$, and multiplication by a scalar as $\alpha x = (\alpha x_1, \alpha x_2, \cdots, \alpha x_N)$. Then $\mathbb{R}^N$ is a linear space over $\mathbb{R}$, and $\mathbb{C}^N$ a linear space over $\mathbb{C}$. ∎

A *subspace* U of a linear space X is a subset of X that is itself a linear space over the same field as X. To check whether a subset U is a subspace it is necessary to verify *closure* (i.e., whether $x \in U$ and $y \in U$ implies $x + y \in U$ and $x \in U$, $\alpha \in \mathcal{F}$ implies $\alpha x \in U$).

B.3. Example: Span. Let $x_1, x_2, \cdots, x_K$ be elements of $\mathbb{C}^N$. The set $\{x_1, x_2, \cdots, x_K\}$ is not a subspace of $\mathbb{C}^N$, but it may be made into one by taking all linear combinations of the vectors $x_1, x_2, \cdots, x_K$. We thus obtain the set

$$\text{span}\,(x_1, x_2, \cdots, x_K) := \left\{x \in \mathbb{C}^N \mid (\exists \alpha_1, \alpha_2, \cdots, \alpha_K \in \mathbb{C}) \quad x = \sum_{i=1}^{K} \alpha_i x_i\right\},$$

which is a subspace. It is called the subspace that is *spanned* by the vectors $x_1, x_2, \cdots, x_K$. This technique of generating subspaces is quite general. ∎

The reason why we are interested in linear spaces is that all the sets of complex-valued time signals we deal with constitute linear spaces.

B.4. Example: The linear signal space $\mathbb{C}^{\mathbb{T}}$. The triple $(\mathbb{C}^{\mathbb{T}}, +, \cdot)$, with $\mathbb{T}$ a time axis, addition defined pointwise as in 2.3.14 and multiplication by a scalar likewise defined pointwise, is a linear space over the complex numbers. In particular, the signal sets $\ell_{\underline{N}}$, ℓ_+, ℓ, $\ell_{\underline{N}}(T)$, $\ell_+(T)$, $\ell(T)$, $\mathcal{L}[a, b]$, $\mathcal{L}_+$, and $\mathcal{L}$ defined in Section 2.2 all are linear spaces.

Norms

The *norm* is a measure for the "size" of a signal.

B.5. Definition: Norm and normed space. Let X be a linear space over the field $\mathcal{F}$ of real or complex scalars. A function

$$\|\cdot\|: X \to \mathbb{R}$$

that maps X into $\mathbb{R}$ is called a *norm* on X if it satisfies the following conditions:

(a) $\|x\| > 0$ for all $x \in X$ (nonnegativity),
(b) $\|x\| = 0$ if and only if $x = 0$ (positive-definiteness),
(c) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathcal{F}$ and all $x \in X$ (homogeneity with respect to scaling),
(d) $\|x + y\| \leq \|x\| + \|y\|$ for all $x \in X$ and $y \in X$ (triangle inequality).

In (c), $|\lambda|$ denotes the absolute value of the scalar λ. The pair $(X, \|\cdot\|)$ is called a *normed* linear space. ∎

Fig. B.1 illustrates the norm and the triangle inequality geometrically.

The p-norm on the signal spaces ℓ and $\mathcal{L}$ is defined in 2.4.3.

B.6. Summary: Normed signal spaces. The signal spaces ℓ_p and $\mathcal{L}_p$ are normed linear subspaces of ℓ and $\mathcal{L}$. ∎

B.7. Exercise: Proof of B.6. Prove B.6. *Hint.* To prove that ℓ_∞ is a subspace of ℓ and $\mathcal{L}_\infty$ a subspace of $\mathcal{L}$ is straightforward, by using the triangle equality for

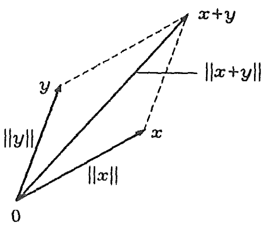


Figure B.1. Norm and the triangle inequality.

complex numbers. To prove that ℓ_2 and ℓ_1 are subspaces of ℓ, use *Minkowski's inequality for sums*. This inequality states that for any $x = (\cdots, x_{-1}, x_0, x_1, x_2, \cdots) \in \ell$ and $y = (\cdots, y_{-1}, y_0, y_1, y_2, \cdots) \in \ell$ and for $1 \leq p < \infty$,

$$\left(\sum_{i=-\infty}^{\infty} |x_i \pm y_i|^p\right)^{1/p} \leq \left(\sum_{i=-\infty}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=-\infty}^{\infty} |y_i|^p\right)^{1/p}.$$

To prove that $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspaces of $\mathcal{L}$, use Minkowski's inequality for *integrals*, which takes the form

$$\left(\int_{-\infty}^{\infty} |x(t) \pm y(t)|^p \, dt\right)^{1/p} \leq \left(\int_{-\infty}^{\infty} |x(t)|^p \, dt\right)^{1/p} + \left(\int_{-\infty}^{\infty} |y(t)|^p \, dt\right)^{1/p}$$

for any $x, y \in \mathcal{L}$, and any $1 \leq p < \infty$. ■

Inner Product

The inner product of two elements of a linear space is axiomatically defined as follows.

B.8. Definition: Inner product. Let X be a linear space over the field $\mathcal{F}$ of real or complex numbers. Then a map

$$\langle \cdot, \cdot \rangle : X \times X \to \mathcal{F}$$

is called an *inner product* on X if it has the following properties.

(a) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in X$. The overbar denotes the complex conjugate (conjugate symmetry),
(b) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and all $\alpha, \beta \in \mathcal{F}$ (bilinearity),
(c) $\langle x, x \rangle$ is real and nonnegative for all $x \in X$,
(d) $\langle x, x \rangle = 0$ if and only if $x = 0$.

A linear space X that has an inner product is called an *inner product space*. ■

The Cauchy-Schwarz inequality plays an important role.

B.9. Summary: The Cauchy-Schwarz inequality. Let X be an inner produ space over the real or complex field $\mathcal{F}$. Then, for any $x, y \in X$

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle.$$

Equality holds if and only if there exists an $\alpha \in \mathcal{F}$ such that $x = \alpha y$ or $y = \alpha x$.

B.10. Proof. If y is the zero element of X then the theorem holds trivially. I y is not the zero element then for any scalar $\alpha \in \mathcal{F}$ it follows by B.8(c) and repeated application of B.8(a) and (b) that

$$\begin{aligned} 0 &\leq \langle x - \alpha y, x - \alpha y \rangle \\ &= \langle x, x \rangle - \alpha \overline{\langle x, y \rangle} - \overline{\alpha} \langle x, y \rangle + |\alpha|^2 \langle y, y \rangle. \end{aligned}$$

If in particular $\alpha = \langle x, y \rangle / \langle y, y \rangle$, the inequality to be proven follows immediately. The remainder of the proof is left to the reader. ■

Inner product spaces automatically have a norm.

B.11. Summary: Natural norm. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Then the map $\|\cdot\| : X \to \mathbb{R}$ defined by

$$\|x\| = \langle x, x \rangle^{1/2}$$

is a norm on X, called the *natural norm*. ■

B.12. Exercise: Proof of B.11. Prove B.11. ■

Supplement C

An Introduction to the Theory of Generalized Signals

In this supplement we present a brief outline of *distribution theory,* which is the mathematical foundation of the theory of generalized functions. *Regular* functions f with domain $\mathbb{R}$ and range $\mathbb{C}$ are specified *pointwise* (i.e., given $t \in \mathbb{R}$ the value $f(t)$ of f at t is well-defined). It turns out that regular functions cannot describe physical phenomena that occur "instantaneously," such as charging a capacitor with wires without resistance or notions such as point masses and charges. This encourages enlarging the set of functions so that they include *singular* functions. These are "functions" that cannot be defined pointwise but only *indirectly,* by specifying their "effect" on a set of *test functions.* Together the regular and singular functions form the *generalized* functions.

Distributions

Distribution theory deals with regular and singular functions in a unified way. The starting point is the observation that a regular function alternatively may be characterized as a *linear functional,* which specifies the effect of the function on a well-defined set of test functions. The set of all linear functionals also includes functionals that do *not* correspond to regular functions. By definition, such functionals correspond to *singular* functions.

The functionals that are considered are called *distributions,* and this is why the investigation of singular functions amounts to distribution theory. Operations on distributions, such as addition, multiplication by a scalar, and differentiation, are defined in such a way that for *regular* distributions, that is, distributions corresponding to regular functions, the operations are equivalent to the conventional operations on the functions.

We start our brief review of distribution theory by explaining what are linear functionals.

C.1. Definition: Linear functionals. Let X be a linear space over the field $\mathbb{C}$. Then a *linear functional* $\mathfrak{f}$ on X is a map $\mathfrak{f}: X \to \mathbb{C}$ such that

$$\mathfrak{f}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \mathfrak{f}(x_1) + \alpha_2 \mathfrak{f}(x_2)$$

for all $\alpha_1, \alpha_2 \in \mathbb{C}$ and all $x_1, x_2 \in X$. ∎

We illustrate the definition with an example.

C.2. Example: Linear functional on $\mathbb{C}^N$. Consider the functional $\mathfrak{f}$ defined on $\mathbb{C}^N$ by

$$\mathfrak{f}(x) = \sum_{i=1}^{N} \alpha_i x_i \qquad \text{for any} \quad x = (x_1, x_2, \cdots, x_N) \in \mathbb{C}^N,$$

with $\alpha_1, \alpha_2, \cdots, \alpha_N$ fixed complex numbers. It is easy to verify that $\mathfrak{f}$ is a linear functional over the field $\mathbb{C}$. ∎

Generalized functions are described by means of linear functionals on the *set of test functions* $\mathfrak{D}$, which is defined as follows.

C.3. Definition: The set of test functions $\mathfrak{D}$. The *set of test functions* $\mathfrak{D}$ consists of all functions in $\mathcal{L}$ that are zero outside some finite interval and may be differentiated arbitrarily often. ∎

A function ϕ whose derivative $D^k\phi$ exists for every $k \in \mathbb{Z}_+$ is called *smooth.*

C.4. Example: Test function. An example of a test function is the function $\phi \in \mathfrak{D}$ given by

$$\phi(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & \text{for } |t| < 1, \\ 0 & \text{for } |t| \geq 1, \end{cases}$$

as shown in Fig. C.1. ∎

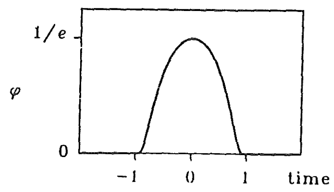


Figure C.1. Example of a test function $\phi \in \mathfrak{D}$.

Distributions are now defined as follows.

C.5. Definition: Distribution. A *distribution* is a linear functional on the set of test functions $\mathfrak{D}$. ■

More precisely, a distribution is a *continuous* linear functional on $\mathfrak{D}$, but we do not dwell on this point.

In the sequel we denote distributions by bold English or Greek characters, usually lower case. In the following examples two typical instances of distributions are considered.

C.6. Example: A distribution. The functional $\mathbf{f}$ on $\mathfrak{D}$, defined by

$$\mathbf{f}(\phi) = \int_a^b \phi(t)\,dt, \qquad \phi \in \mathfrak{D},$$

with a and b fixed real numbers, is linear. Hence $\mathbf{f}$ is a distribution. ■

C.7. Definition: Delta distribution. The linear functional $\boldsymbol{\delta}$ on $\mathfrak{D}$, defined by

$$\boldsymbol{\delta}(\phi) = \phi(0), \qquad \phi \in \mathfrak{D},$$

is a distribution, called the *delta distribution*. ■

By way of example, the value of $\boldsymbol{\delta}$ at the function ϕ given in C.4 is

$$\boldsymbol{\delta}(\phi) = e^{-\frac{1}{1-t^2}}\Big|_{t=0} = e^{-1}.$$

The distributions of C.6 and C.7 differ in one essential respect. The distribution of C.6 is of the form

$$\mathbf{f}(\phi) = \int_{-\infty}^{\infty} f(t)\phi(t)\,dt, \qquad \phi \in \mathfrak{D}, \tag{1}$$

where the function f is given by

$$f(t) = \begin{cases} 1 & \text{for } a \le t < b, \\ 0 & \text{otherwise.} \end{cases}$$

The delta distribution of C.7, however, *cannot* be represented in the form (1). Apparently some but not all distributions may be represented in the integral form (1) This introduces the following dichotomy among distributions.

C.8. Definition: Regular and singular distributions. A distribution $\mathbf{f}$ is called *regular* if there exists a function $f \in \mathcal{L}$ such that

$$\mathbf{f}(\phi) = \int_{-\infty}^{\infty} f(t)\phi(t)\,dt, \qquad \phi \in \mathfrak{D}.$$

Otherwise, $\mathbf{f}$ is called *singular*. If $\mathbf{f}$ is regular, then we say that the regular function $f \in \mathcal{L}$ *represents* the distribution. ■

As was pointed out before, distributions are denoted by bold characters. The function that *represents* a distribution is usually denoted by the same character but in Italic typeface. Although if a distribution $\mathbf{f}$ is singular, there exists *no* function $f \in \mathcal{L}$ such that

$$\mathbf{f}(\phi) = \int_{-\infty}^{\infty} f(t)\phi(t)\,dt, \qquad \phi \in \mathfrak{D},$$

we nevertheless often write $\mathbf{f}$ *symbolically* as an integral and call the "function" f that enters into it a *singular* function. The singular functions together with the regular functions are called *generalized* functions.

We thus write the delta distribution $\boldsymbol{\delta}$ of C.7 symbolically as

$$\boldsymbol{\delta}(\phi) = \int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt = \phi(0), \qquad \phi \in \mathfrak{D}.$$

The singular function δ is called the *delta function* or *δ-function*.

C.9. Remark: Operating with the δ-distribution on a function not in $\mathfrak{D}$. Because the outcome of $\boldsymbol{\delta}(\phi)$ only depends on the value of ϕ at 0, we may also operate with the δ-function on functions ϕ that are not in $\mathfrak{D}$ (i.e., are not zero outside some finite interval and are not smooth). When applying the δ-function, the only requirement is that ϕ be continuous at 0. We thus have

$$\int_{-\infty}^{\infty} \delta(t)f(t)\,dt = f(0)$$

for any $f \in \mathcal{L}$ that is continuous at 0. ■

Basic Operations

In what follows we extend the various operations on and among regular time signals, such as addition, scaling, time translation, and differentiation to distributions and generalized signals, taking care, of course, that these extensions are *consistent* with the definitions for regular signals. For each new definition we first verify consistency. Next we show the effect of the operation on the δ-distribution and -function, and thus generate all the distributions and singular signals needed in this text.

Before proceeding, we define *equality* of two distributions.

C.10. Definition: Equality of distributions. Two distributions $\mathbf{f}$ and $\mathbf{g}$ are *equal* if

$$\mathbf{f}(\phi) = \mathbf{g}(\phi) \qquad \text{for all} \quad \phi \in \mathcal{D}. \qquad \blacksquare$$

Addition and Multiplication by a Scalar

The definitions of addition of distributions and multiplication of a distribution by a scalar are straightforward.

C.11. Definition. Addition of distributions and multiplication by a scalar. If $\mathbf{f}$ and $\mathbf{g}$ are distributions and α and β scalars in $\mathbb{C}$, then the distribution $\alpha\mathbf{f} + \beta\mathbf{g}$ is defined by

$$(\alpha\mathbf{f} + \beta\mathbf{g})(\phi) = \alpha\mathbf{f}(\phi) + \beta\mathbf{g}(\phi), \qquad \phi \in \mathcal{D}.$$

If $\mathbf{f}$ and $\mathbf{g}$ are represented by the regular *or* singular functions f and g, respectively, then we write $\alpha f + \beta g$ for the generalized function that represents $\alpha\mathbf{f} + \beta\mathbf{g}$. ■

Note that defining a distribution means specifying its value for each test function ϕ in the set of test functions $\mathcal{D}$.

C.12. Consistency proof: Addition and multiplication by a scalar. If $\mathbf{f}$ and $\mathbf{g}$ are *regular* distributions represented by the regular functions $f \in \mathcal{L}$ and $g \in \mathcal{L}$, respectively, then the distribution $\alpha\mathbf{f} + \beta\mathbf{g}$ as defined in C.11 is a regular distribution represented by $\alpha f + \beta g \in \mathcal{L}$. This is easy to see because

$$\begin{aligned}(\alpha\mathbf{f} + \beta\mathbf{g})(\phi) &= \alpha\mathbf{f}(\phi) + \beta\mathbf{g}(\phi)\\ &= \alpha\int_{-\infty}^{\infty} f(t)\phi(t)\,dt + \beta\int_{-\infty}^{\infty} g(t)\phi(t)\,dt\\ &= \int_{-\infty}^{\infty} (\alpha f + \beta g)(t)\phi(t)\,dt, \qquad \phi \in \mathcal{D}. \qquad \blacksquare\end{aligned}$$

Now that addition and multiplication have been defined, it is easy to see that the set of all distributions forms a *linear space*. Correspondingly, also the set of generalized functions, consisting of the set of regular functions $\mathcal{L}$ together with the singular functions, is a linear space. Abusing notation, we sometimes denote this space as the continuous-time signal space $\mathcal{L}$. Signals in this space that are equal in the sense of distributions are considered to be one and the same signal.

Time Scaling and Time Translation

Time scaling of continuous-time signals may be defined by the *time scaling operator* μ^α, with α a nonzero real number, that transforms any signal x in $\mathcal{L}$ to the signal $\mu^\alpha x$ in $\mathcal{L}$, given by

$$(\mu^\alpha x)(t) = x(\alpha t), \qquad t \in \mathbb{R}.$$

This leads to the following definition of time scaling for distributions.

C.13. Definition: Time scaling of distributions. *Time scaling* the distribution $\mathbf{f}$ by the nonzero real constant α results in the distribution $\mu^\alpha\mathbf{f}$, defined by

$$(\mu^\alpha\mathbf{f})(\phi) = \frac{1}{|\alpha|}\mathbf{f}(\mu^{1/\alpha}\phi) \qquad \text{for all} \quad \phi \in \mathcal{D}.$$

If $\mathbf{f}$ is represented by the regular *or* singular function f, then we write $\mu^\alpha f$, or $f(\alpha t)$, $t \in \mathbb{R}$, for the generalized function that represents $\mu^\alpha\mathbf{f}$. ■

C.14. Consistency proof: Time scaling. If $\mathbf{f}$ is a regular distribution represented by the function f, time scaling $\mathbf{f}$ by α according to C.13 results in the distribution $\mu^\alpha\mathbf{f}$ given by

$$(\mu^\alpha\mathbf{f})(\phi) = \frac{1}{|\alpha|}\int_{-\infty}^{\infty} f(t)(\mu^{1/\alpha}\phi)(t)\,dt = \frac{1}{|\alpha|}\int_{-\infty}^{\infty} f(t)\phi(t/\alpha)\,dt$$

for all $\phi \in \mathcal{D}$. The change of variable $t/\alpha = \tau$ results in

$$(\mu^\alpha\mathbf{f})(\phi) = \int_{-\infty}^{\infty} f(\alpha\tau)\phi(\tau)\,d\tau = \int_{-\infty}^{\infty} (\mu^\alpha f)(\tau)\phi(\tau)\,d\tau$$

for all $\phi \in \mathcal{D}$. This proves that $\mu^\alpha\mathbf{f}$ is represented by $\mu^\alpha f$, as required for consistency. ■

C.15. Example: Time scaling of the δ-function. Let us see what time scaling by α does to the delta distribution and the δ-function. Application of the definition of a time scaled distribution yields

$$(\mu^\alpha \delta)(\phi) = \frac{1}{|\alpha|}\delta(\mu^{1/\alpha}\phi) = \frac{1}{|\alpha|}(\mu^{1/\alpha}\phi)(0) = \frac{1}{|\alpha|}\phi(0/\alpha)$$
$$= \frac{1}{|\alpha|}\phi(0)$$

for all $\phi \in \mathfrak{D}$. It follows that

$$\mu^\alpha \delta = \frac{1}{|\alpha|}\delta.$$

In terms of generalized functions,

$$\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t), \qquad t \in \mathbb{R}.$$ ■

C.16. Exercise: Time reversing a δ-function. Show that $\delta(-t) = \delta(t)$, $t \in \mathbb{R}$. ■

Time translation of continuous-time signals may be defined by the *back shift operator* σ^θ, with θ a real number, that transforms any time signal x in $\mathcal{L}$ to the back shifted signal $\sigma^\theta x$ in $\mathcal{L}$, given by

$$(\sigma^\theta x)(t) = x(t + \theta), \qquad t \in \mathbb{R}.$$

Time translation of a distribution is defined as follows.

C.17. Definition: Time translation of a distribution. *Time translating* the distribution **f** by the real number θ results in the *back shifted* distribution $\sigma^\theta\mathbf{f}$, defined by

$$(\sigma^\theta \mathbf{f})(\phi) = \mathbf{f}(\sigma^{-\theta}\phi) \qquad \text{for all} \quad \phi \in \mathfrak{D}.$$

If **f** is represented by the regular or singular function f, then we write $\sigma^\theta f$, or $f(t + \theta)$, $t \in \mathbb{R}$, for the generalized function that represents $\sigma^\theta\mathbf{f}$. ■

C.18. Consistency proof: Time translation of a distribution. We verify that back shifting a regular distribution **f** represented by the regular function f by θ results in a regular distribution $\sigma^\theta\mathbf{f}$ represented by the back shifted function $\sigma^\theta f$. Indeed, if **f** is regular it follows from the definition of time translation that

$$(\sigma^\theta \mathbf{f})(\phi) = \mathbf{f}(\sigma^{-\theta}\phi) = \int_{-\infty}^{\infty} f(t)(\sigma^{-\theta}\phi)(t)\,dt = \int_{-\infty}^{\infty} f(t)\phi(t - \theta)\,dt$$

for all $\phi \in \mathfrak{D}$. By the change of variable $t - \theta = \tau$ it follows that

$$(\sigma^\theta \mathbf{f})(\phi) = \int_{-\infty}^{\infty} f(\tau + \theta)\phi(\tau)\,d\tau = \int_{-\infty}^{\infty} (\sigma^\theta f)(\tau)\phi(\tau)\,d\tau$$

for all $\phi \in \mathfrak{D}$, which proves that $\sigma^\theta\mathbf{f}$ is represented by $\sigma^\theta f$. ■

C.19. Example: Time translation of a δ-function. Let us see what happens when the delta distribution and the δ-function are translated. By the definition of time translation we have

$$(\sigma^\theta \delta)(\phi) = \delta(\sigma^{-\theta}\phi) = (\sigma^{-\theta}\phi)(0) = \phi(-\theta)$$

for all $\phi \in \mathfrak{D}$. In terms of generalized functions,

$$\int_{-\infty}^{\infty} \delta(t + \theta)\phi(t)\,dt = \phi(-\theta).$$

The formula shows that the δ-function that is back shifted by θ singles out the value of the test function ϕ at time $-\theta$ rather than at time 0. ■

Multiplication by a Function

The *product* of two distributions, in general, is not defined. An exception occurs when one of the distributions is regular and represented by a *smooth* function (i.e., a function that may be differentiated as often as desired).

C.20. Definition: Multiplication of a distribution by a smooth function. If **f** is a distribution, and **g** is a regular distribution represented by a smooth regular function g, then the product **fg** is the distribution defined by

$$(\mathbf{fg})(\phi) = \mathbf{f}(g\phi) \qquad \text{for all} \quad \phi \in \mathfrak{D}.$$

If **f** is represented by the regular or singular function f, then we write fg for the generalized function that represents **fg**. ■

C.21. Consistency proof: Product of a distribution and a smooth function. We verify the consistency of the definition by letting **f** be a regular distribution represented by the regular function f. Then if the distribution **g** is regular and represented by the smooth function g, by C.20 the product **fg** is defined by

$$(\mathbf{fg})(\phi) = \mathbf{f}(g\phi) = \int_{-\infty}^{\infty} f(t)[g(t)\phi(t)]\,dt = \int_{-\infty}^{\infty} [f(t)g(t)]\phi(t)\,dt$$

for all $\phi \in \mathfrak{D}$, which confirms that **fg** is represented by fg. ■

C.22. Example: Product of the δ-function and a smooth function. Suppose that $\mathbf{f}$ is the delta distribution. Then, if $\mathbf{g}$ is regular and represented by the smooth function g,

$$(\delta\mathbf{g})(\phi) = \delta(g\phi) = g(0)\phi(0) = g(0)\delta(\phi)$$

for all $\phi \in \mathfrak{D}$. It follows that $\delta\mathbf{g} = g(0)\delta$. In terms of generalized functions,

$$g(t)\delta(t) = g(0)\delta(t), \qquad t \in \mathbb{R}.$$

Note that the product $g\delta$ only depends on the value $g(0)$ of g at 0. ∎

Differentiation

We next discuss *differentiation* of distributions. A striking property of distributions and generalized functions is that they *always* have derivatives of *any* order.

C.23. Definition: Differentiation of distributions. The *derivative* of a distribution $\mathbf{f}$ is denoted $D\mathbf{f}$, $\mathbf{f}'$ or $\mathbf{f}^{(1)}$, and defined by

$$(D\mathbf{f})(\phi) = -\mathbf{f}(D\phi) \qquad \text{for all} \quad \phi \in \mathfrak{D}.$$

If $\mathbf{f}$ is represented by the regular or singular function f, then we write Df, f' or $f^{(1)}$ for the generalized function that represents $D\mathbf{f}$. ∎

It follows from Definition C.23 that the derivative $D\mathbf{f}$ of a distribution always exists. Repeated application of the definition leads to the *nth derivative* $D^n\mathbf{f}$ of $\mathbf{f}$, with n any nonnegative integer, given by

$$(D^n\mathbf{f})(\phi) = (-1)^n\mathbf{f}(D^n\phi) \qquad \text{for all} \quad \phi \in \mathfrak{D}.$$

We sometimes denote $D^n\mathbf{f}$ by $\mathbf{f}^{(n)}$.

C.24. Consistency proof: Differentiation of a distribution. As usual, we verify that the definition reduces to the conventional notion in case $\mathbf{f}$ is a regular distribution, represented by a differentiable regular function f. It follows that

$$(D\mathbf{f})(\phi) = -\mathbf{f}(D\phi) = -\int_{-\infty}^{\infty} f(t)(D\phi)(t)\, dt = -\int_{-\infty}^{\infty} f(t)\frac{d\phi(t)}{dt}\, dt$$

for all $\phi \in \mathfrak{D}$. Integration by parts results in

$$(D\mathbf{f})(\phi) = -f(t)\phi(t)\Big|_{t=-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{df(t)}{dt}\phi(t)\, dt$$
$$= \int_{-\infty}^{\infty} \frac{df(t)}{dt}\phi(t)\, dt$$

for all $\phi \in \mathfrak{D}$. Here we used the fact that ϕ is zero outside some finite interval so that $\phi(-\infty) = \phi(\infty) = 0$. It follows that $D\mathbf{f}$ is represented by Df, as required. ∎

An interesting application of the differentiation formula is that the unit step has a derivative, in the distribution sense, which is the delta function.

C.25. Example: The δ-function is the derivative of the unit step. The unit step is the distribution $\mathbf{1}$ defined by

$$\mathbf{1}(\phi) = \int_{-\infty}^{\infty} \mathbb{1}(t)\phi(t)\, dt = \int_{0}^{\infty} \phi(t)\, dt \qquad \text{for all} \quad \phi \in \mathfrak{D}.$$

Following C.23, the derivative of this distribution is the distribution defined by

$$(D\mathbf{1})(\phi) = -\mathbf{1}(D\phi) = -\int_{0}^{\infty} (D\phi)(t)\, dt = -\int_{0}^{\infty} \frac{d\phi(t)}{dt}\, dt = \phi(0)$$
$$= \delta(\phi)$$

for all $\phi \in \mathfrak{D}$. As a result, $D\mathbf{1} = \delta$. In terms of generalized functions we have

$$\frac{d}{dt}\mathbb{1}(t) = \delta(t), \qquad t \in \mathbb{R}.$$ ∎

Definition C.24 allows differentiating δ-functions as often as desired.

C.26. Example: The nth derivative of the delta distribution and the δ-function. The nth derivative of the delta distribution is defined by

$$(D^n\delta)(\phi) = (-1)^n\delta(D^n\phi) = (-1)^n\frac{d^n\phi(t)}{dt^n}\Big|_{t=0}$$

for all $\phi \in \mathfrak{D}$. We usually write the nth derivative of the δ-distribution as

$$D^n\delta = \delta^{(n)},$$

and represent it by the singular function $\delta^{(n)}$. In terms of generalized functions we have

$$\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t)\,dt = (-1)^n \frac{d^n\phi(t)}{dt^n}\bigg|_{t=0}. \qquad \blacksquare$$

C.27. Exercise. Product of $\delta^{(n)}$ and a smooth function.

(a) Show that if **g** is a regular distribution, represented by a smooth regular function g,

$$\mathbf{g}\boldsymbol{\delta}' = g(0)\boldsymbol{\delta}' - g'(0)\boldsymbol{\delta}.$$

In terms of generalized functions,

$$g(t)\delta'(t) = g(0)\delta'(t) - g'(0)\delta(t), \qquad t \in \mathbb{R}.$$

(b) Show by induction that if **g** is a regular distribution represented by the smooth function g,

$$\mathbf{g}\boldsymbol{\delta}^{(n)} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g^{(k)}(0)\boldsymbol{\delta}^{(n-k)},$$

or, in terms of generalized functions,

$$g(t)\delta^{(n)}(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g^{(k)}(0)\delta^{(n-k)}(t), \qquad t \in \mathbb{R}. \qquad \blacksquare$$

The converse of differentiation is of course integration.

C.28. Definition: Indefinite integral of distributions. If **f** is a distribution, then any distribution **F** such that $D\mathbf{F} = \mathbf{f}$ is called an *indefinite integral* of **f**. ■

C.29. Exercise: Indefinite integral.

(a) Suppose that **f** is a regular distribution represented by the regular function f, and let F be an indefinite integral (in the regular sense) of f. Show that the distribution **F** represented by F is an indefinite integral of the distribution **f**.

(b) Prove that if **F** is an indefinite integral of **f**, also $\mathbf{c} + \mathbf{F}$, with **c** the regular distribution represented by the constant $c \in \mathbb{C}$, is an indefinite integral of **f**.

(c) Show that the unit step $\mathbb{1} \in \mathcal{L}$ is an indefinite integral of the δ-function. ■

Infinite Sums

By scaling, adding, shifting, and differentiating δ-functions, a wide variety of generalized functions may be obtained, which suit most of our purposes in this text, provided also *infinite* sums are allowed. In particular we want to consider signals such as the "infinite comb," given by

$$w_P(t) = \sum_{n=-\infty}^{\infty} \delta(t + nP), \qquad t \in \mathbb{R},$$

and depicted in Fig. C.2. To handle this we need define the *convergence* of a sequence of distributions.

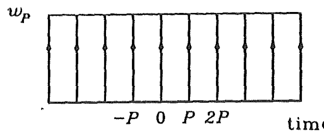


Figure C.2. The infinite comb.

C.30. Definition: Convergence of a sequence of distributions and of an infinite sum of distributions.

(a) *Convergence.* A sequence of distributions $\mathbf{f}_n$, $n = 1, 2, 3, \cdots$, *converges* to a distribution **f** if $\mathbf{f}_n(\phi) \to \mathbf{f}(\phi)$ as $n \to \infty$ for any $\phi \in \mathfrak{D}$. If a sequence $\mathbf{f}_n$, $n = 1, 2, 3, \cdots$, converges to **f**, then we write

$$\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}.$$

(b) *Infinite sums.* The infinite sum

$$\sum_{n=1}^{\infty} \mathbf{f}_n$$

is said to exist if the sequence of partial sums

$$\mathbf{s}_N = \sum_{n=1}^{N} \mathbf{f}_n, \qquad N = 1, 2, \cdots,$$

converges as $N \to \infty$. In this case

$$\sum_{n=1}^{\infty} \mathbf{f}_n = \mathbf{s},$$

where $\mathbf{s} = \lim_{N\to\infty} \mathbf{s}_N$. ■

C.31. Example: Infinite comb. The infinite comb

$$\mathbf{w}_P = \sum_{n=-\infty}^{\infty} \sigma^{nP}\boldsymbol{\delta},$$

with P a real number, is a well-defined distribution, because the sequence of partial sums

$$\sum_{n=-N}^{N} (\sigma^{nP}\delta)(\phi) = \sum_{n=-N}^{N} \delta(\sigma^{-nP}\phi) = \sum_{n=-N}^{N} \phi(-nP)$$

converges as $N \to \infty$ for any $\phi \in \mathfrak{D}$. In fact, because ϕ is zero outside some finite interval, the sum only has a *finite* number of nonzero terms. In terms of generalized functions we may represent the infinite comb as

$$w_P(t) = \sum_{n=-\infty}^{\infty} \delta(t + nP), \qquad t \in \mathbb{R}.$$ ∎

What follows has important applications.

C.32. Summary: Convergence of a sequence of derivatives and term-by-term differentiation and integration.
(a) *Convergence of a sequence of derivatives.* Suppose that the sequence of distributions $\mathbf{f}_n$, $n = 1, 2, 3, \cdots$, converges to a distribution $\mathbf{f}$. Then the sequence $D\mathbf{f}_n$, $n = 1, 2, 3, \cdots$, converges to $D\mathbf{f}$.

(b) *Term-by-term differentiation and integration.* Suppose that

$$\mathbf{f} = \sum_{n=1}^{\infty} \mathbf{f}_n$$

exists. Then,

$$D\mathbf{f} = \sum_{n=1}^{\infty} D\mathbf{f}_n.$$

Likewise, if $\mathbf{F}_n$ is an indefinite integral of $\mathbf{f}_n$ for $n = 1, 2, 3, \cdots$, and

$$\mathbf{F} = \sum_{n=1}^{\infty} \mathbf{F}_n$$

exists, then $\mathbf{F}$ is an indefinite integral of $\mathbf{f}$. ∎

C.33. Example: Comb of derivatives. The comb $\mathbf{w}_P^{(k)}$ of kth order derivatives of delta distributions may be obtained by differentiating the infinite comb $\mathbf{w}_P$ term by term, that is,

$$\mathbf{w}_P^{(k)} = D^k\left(\sum_{n=-\infty}^{\infty} \sigma^{nP}\boldsymbol{\delta}\right) = \sum_{n=-\infty}^{\infty} D^k\sigma^{nP}\boldsymbol{\delta} = \sum_{n=-\infty}^{\infty} \sigma^{nP}D^k\boldsymbol{\delta}$$
$$= \sum_{n=-\infty}^{\infty} \sigma^{nP}\boldsymbol{\delta}^{(k)}.$$

In generalized function form,

$$w_P^{(k)}(t) = \sum_{n=-\infty}^{\infty} \delta^{(k)}(t - nP), \qquad t \in \mathbb{R}.$$ ∎

C.34. Exercise: Indefinite integral of the infinite comb. Let int be the entier function defined in 2.3.4. Show that the function W_P defined by

$$W_P(t) = \text{int}\,(t/P), \qquad t \in \mathbb{R},$$

is an indefinite integral of the infinite comb w_P. ∎

Periodic Distributions

The graph of the infinite comb of Fig. C.2 has a periodic appearance. Periodicity of distributions is defined as follows.

C.35. Definition: Periodic distribution.
(a) *Periodic distribution.* The distribution $\mathbf{f}$ is *periodic* if $\sigma^P\mathbf{f} = \mathbf{f}$ for some $P \in \mathbb{R}$ with $P \neq 0$.

(b) *Period of a periodic distribution.* The periodic distribution $\mathbf{f}$ has *period* P if P is the smallest positive number such that $\sigma^P\mathbf{f} = \mathbf{f}$. ∎

C.36. Exercise: Consistency proof of periodicity of distributions. Prove that the regular distribution $\mathbf{f}$ represented by the regular function f is periodic if and only if f is periodic. Show also that the period of $\mathbf{f}$ equals the period of f. ∎

C.37. Example: The infinite comb. The infinite comb $\mathbf{w}_P$ of Example C.31 is periodic since

$$\sigma^P\mathbf{w}_P = \sigma^P\left(\sum_{n=-\infty}^{\infty} \sigma^{nP}\boldsymbol{\delta}\right) = \sum_{n=-\infty}^{\infty} \sigma^{(n+1)P}\boldsymbol{\delta} = \sum_{k=-\infty}^{\infty} \sigma^{kP}\boldsymbol{\delta} = \mathbf{w}_P.$$

Here we use the fact that time translation of infinite sums may be done term-by-term, and substituted $k = n + 1$. The period of $\mathbf{w}_P$ is P. ∎

Convolution

We complement Section 3.5 with a discussion of the *convolution* of distributions and generalized signals. Like the other operations on distributions, convolution of distributions need be defined indirectly.

C.38. Definition: Convolution of distributions. The *convolution* of two distributions $\mathbf{f}$ and $\mathbf{g}$, if it exists, is the distribution $\mathbf{f} * \mathbf{g}$ defined by

$$(\mathbf{f} * \mathbf{g})(\phi) = \mathbf{f}(\psi),$$

for all $\phi \in \mathfrak{D}$, where ψ is given by

$$\psi(t) = \mathbf{g}(\sigma^t\phi), \qquad t \in \mathbb{R}.$$

As before, σ is the back shift operator. If $\mathbf{f}$ and $\mathbf{g}$ are represented by the regular or singular functions f and g, respectively, we write $f * g$ for the generalized function that represents $\mathbf{f} * \mathbf{g}$. ∎

The definition shows that to determine the effect of the convolution $\mathbf{f} * \mathbf{g}$ on a test function ϕ, first the function ψ is obtained pointwise for each $t \in \mathbb{R}$ by applying $\mathbf{g}$ to the back shifted function $\sigma^t\phi$, and then $\mathbf{f}$ is applied to ψ.

C.39. Consistency proof: Convolution. We show that if $\mathbf{f}$ and $\mathbf{g}$ are regular distributions represented by regular functions f and g such that $f * g$ exists, $\mathbf{f} * \mathbf{g}$ is regular and represented by $f * g$. If $\mathbf{g}$ is regular, then the function ψ in C.38 is given by

$$\begin{aligned}\psi(t) = \mathbf{g}(\sigma^t\phi) &= \int_{-\infty}^{\infty} g(\tau)(\sigma^t\phi)(\tau)\, d\tau = \int_{-\infty}^{\infty} g(\tau)\phi(\tau + t)\, d\tau \\ &= \int_{-\infty}^{\infty} g(\theta - t)\phi(\theta)\, d\theta, \qquad t \in \mathbb{R},\end{aligned}$$

where we substituted $\tau + t = \theta$. It may be proved that $\mathbf{f}(\psi)$ is well-defined for any test function ϕ and as a result,

$$\begin{aligned}(\mathbf{f} * \mathbf{g})(\phi) = \mathbf{f}(\psi) &= \int_{-\infty}^{\infty} f(t)\psi(t)\, dt \\ &= \int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty} g(\theta - t)\phi(\theta)\, d\theta\right) dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)g(\theta - t)\, dt\right)\phi(\theta)\, d\theta \\ &= \int_{-\infty}^{\infty} (f * g)(\theta)\phi(\theta)\, d\theta.\end{aligned}$$

This shows that $\mathbf{f} * \mathbf{g}$ is represented by $f * g$. ∎

C.40. Example: The delta function is the unit of the convolution. Suppose that $\mathbf{f}$ is any distribution, and let $\mathbf{g}$ be the delta distribution. Then, for any $\phi \in \mathfrak{D}$ the function ψ is given by

$$\psi(t) = \boldsymbol{\delta}(\sigma^t\phi) = (\sigma^t\phi)(0) = \phi(t), \qquad t \in \mathbb{R},$$

so that $\psi = \phi$ and, hence, $\psi \in \mathfrak{D}$. It follows that $\mathbf{f} * \boldsymbol{\delta}$ is defined by

$$(\mathbf{f} * \boldsymbol{\delta})(\phi) = \mathbf{f}(\psi) = \mathbf{f}(\phi) \qquad \text{for all} \quad \phi \in \mathfrak{D}.$$

Consequently,

$$\mathbf{f} * \boldsymbol{\delta} = \mathbf{f}$$

for any distribution $\mathbf{f}$. In terms of generalized functions,

$$f * \delta = f$$

for any regular or singular function f. Written in full generalized function notation we have

$$\int_{-\infty}^{\infty} f(t - \tau)\delta(\tau)\, d\tau = \int_{-\infty}^{\infty} \delta(\tau)f(t - \tau)\, d\tau = f(t), \qquad t \in \mathbb{R}.$$

The delta function is what is called the *unit* of the convolution. ∎

On the basis of Definition C.38 of the convolution of distributions and generalized functions, it may be proved that all the properties 3.5.2 of the continuous-time convolution also apply to the generalized convolution.

C.41. Example: Convolution with derivatives of delta functions. We use several of the properties of the convolution to obtain some formulas involving convolution with derivatives of delta functions. Let n and m be nonnegative integers. Then, if f and g are generalized signals, by the differentiation property and the commutativity of the convolution we have

$$D^{n+m}(f * g) = D^n(D^m(f * g)) = D^n(f * (D^m g)) = (D^n f) * (D^m g).$$

In particular, since $f * \delta = f$, it follows that

$$f * \delta^{(n)} = D^n f,$$

for all n in $\mathbb{Z}_+$ and any generalized signal f. Also, since $\delta * \delta = \delta$,

$$\delta^{(n)} * \delta^{(m)} = \delta^{(n+m)}$$

for all n and m in $\mathbb{Z}_+$. In full generalized signal notation we have

$$\int_{-\infty}^{\infty} f(t-\tau)\delta^{(n)}(\tau)\,d\tau = \frac{d^n}{dt^n}f(t), \qquad t \in \mathbb{R},$$

for every generalized signal f, and

$$\int_{-\infty}^{\infty} \delta^{(n)}(t-\tau)\delta^{(m)}(\tau)\,d\tau = \delta^{(n+m)}(t), \qquad t \in \mathbb{R},$$

for all nonnegative n and m. ∎

C.42. Exercise: Integration via convolution. The convolution provides a way to obtain the integral of a distribution.

(a) Let $\mathbf{1}$ be the distribution represented by the unit step $\mathbb{1}$ and $\mathbf{c}$ the regular distribution represented by the constant $c \in \mathbb{C}$. Prove that if

$$\mathbf{F} = \mathbf{1} * \mathbf{f} + \mathbf{c}$$

exists, it is an indefinite integral of $\mathbf{f}$ for any $\mathbf{c}$.

(b) Suppose that f is a regular function such that the definite integral

$$F(t) = \int_{-\infty}^{t} f(\tau)\,d\tau, \qquad t \in \mathbb{R}, \tag{2}$$

exists for all $t \in \mathbb{R}$. Prove that F represents the distribution $\mathbf{1} * \mathbf{f}$.

(c) By analogy, we write the generalized function F that represents the definite integral $\mathbf{F} = \mathbf{1} * \mathbf{f}$, if it exists, in function form as (2). Prove that

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \mathbb{1}(t), \qquad t \in \mathbb{R}.$$ ∎

C.43. Remark: Existence of the convolution. The convolution of two distributions may or may not exist, just as the convolution of two regular functions does not always exist. Like for regular convolutions (see 3.4.5) sufficient conditions are available for the existence of the convolution of distributions depending on their supports. A distribution $\mathbf{f}$ has *bounded support* if there exists a bounded interval $[a, b]$ such that $\mathbf{f}(\phi) = 0$ for every test function $\phi \in \mathfrak{D}$ with support *outside* this interval. Similarly, the distribution $\mathbf{f}$ is *right one-sided* if its support is right-one sided (i.e., there exists a semi-infinite interval $[a, \infty)$ such that $\mathbf{f}(\phi) = 0$ for every test function ϕ with support outside this interval). Left one-sided distributions are defined analogously.

If $\mathbf{f}$ and $\mathbf{g}$ are two distributions, then we have:

(a) If $\mathbf{f}$ or $\mathbf{g}$ has bounded support, then $\mathbf{f} * \mathbf{g}$ exists. If $\mathbf{f}$ and $\mathbf{g}$ both have bounded support, then also $\mathbf{f} * \mathbf{g}$ has bounded support.
(b) If $\mathbf{f}$ and $\mathbf{g}$ are both one-sided (both left or both right), then $\mathbf{f} * \mathbf{g}$ exists and is also one-sided (in the same direction as $\mathbf{f}$ and $\mathbf{g}$.)
(c) If $\mathbf{f}$ or $\mathbf{g}$ is regular and represented by a smooth function with bounded support, then $\mathbf{f} * \mathbf{g}$ exists, is regular, and may be represented by a smooth function. ∎

Tempered Signals

In Section 7.4 the CCFT of finite-energy continuous-time signals is studied. There are many useful signals, however, that do not belong to the space $\mathcal{L}_2$, such as the constant signal, the unit step, and the δ-function, and may easily figure as inputs to linear systems. We therefore extend the CCFT to a much wider class of signals than $\mathcal{L}_2$, namely, the signals *of polynomial growth,* also called *tempered* signals. This class includes both singular and regular signals. The constant signal, the unit step, and δ-function belong to it, but signals that increase faster than polynomials, such as exponentially increasing signals, do not.

Regular tempered signals include all continuous-time signals of polynomial growth, that is, signals x for which there exist a nonnegative integer N and real constants β and γ such that

$$|x(t)| \leq \beta|t|^N + \gamma, \qquad t \in \mathbb{R}.$$

Singular tempered signals need be introduced by using the notion of *distribution*. A tempered signal is the generalized signal defining a *tempered distribution*. A tempered distribution is a linear functional on the set of test functions $\mathcal{S}$ of *rapid decay*.

C.44. Definition: Test functions of rapid decay. The set $\mathcal{S}$ of test functions of rapid decay consists of all smooth functions on $\mathbb{R}$ that decay faster than any polynomial, that is, ϕ belongs to $\mathcal{S}$ if it is smooth and there exist a nonnegative number N and a real constant α such that

$$|t^N\phi(t)| \leq \alpha, \qquad t \in \mathbb{R}.$$ ∎

A smooth function of bounded support obviously is of rapid decay and, hence, belongs to $\mathcal{S}$. It follows that the set of test functions $\mathfrak{D}$ of bounded support of C.3 is contained in $\mathcal{S}$.

C.45. Example: Test function of rapid decay. The "Gaussian bell" is the smooth function ϕ defined by

$$\phi(t) = \frac{1}{\sigma\sqrt{2\pi}}e^{-t^2/2\sigma^2}, \qquad t \in \mathbb{R},$$

with σ a positive real number. Because ϕ has unbounded support, it does not belong to $\mathscr{D}$, but it clearly is an element of $\mathscr{S}$. ■

Tempered distributions are defined as linear functionals on the set of test functions $\mathscr{S}$ of rapid decay. *Tempered* generalized signals are the generalized functions that represent tempered distributions.

C.46. Definition: Tempered distributions and tempered generalized signals. A *tempered* distribution is a linear functional on the set of test functions $\mathscr{S}$. The generalized signal representing a tempered distribution is called a *tempered* signal. ■

Because the set of test functions $\mathscr{D}$ of bounded support is contained in the set of test functions $\mathscr{S}$ of rapid decay, every linear functional on $\mathscr{S}$ is also a linear functional on $\mathscr{D}$. As a result, every tempered distribution is a distribution as defined in C.5, and every tempered generalized signal is a generalized signal. The converse is not true: not every generalized signal is tempered.

C.47. Examples: Tempered distributions and signals.

(a) *Regular signal of polynomial growth.* Every regular signal x of polynomial growth defines a tempered distribution $\mathbf{x}$ given by

$$\mathbf{x}(\phi) = \int_{-\infty}^{\infty} x(t)\phi(t)\, dt, \qquad \phi \in \mathscr{S}.$$

(b) *The δ-function and its derivatives.* The delta distribution, defined by

$$\delta(\phi) = \int_{-\infty}^{\infty} \delta(t)\phi(t)\, dt = \phi(0), \qquad \phi \in \mathscr{S},$$

is a tempered distribution. Hence, the δ-function is a tempered singular signal. Similarly, all derivatives of delta functions $\delta^{(k)}$ are tempered singular signals.

(c) *Periodic generalized signals.* Every *periodic* generalized signal is tempered.

(d) *Exponentially increasing signals are not tempered.* Exponentially increasing signals, such as the signal x defined by $x(t) = e^{\alpha t}$, $t \in \mathbb{R}$, with Re $(\alpha) \neq 0$, are generalized signals, but not tempered. ■

The CCFT of Tempered Signals

Preparatory to the definition of the CCFT of signals of polynomial growth we state the following important fact.

C.48. Summary: The CCFT of test functions of rapid decay. The CCFT a bijection from the set of test functions $\mathscr{S}$ of rapid decay to itself.

The statement implies that if ϕ belongs to $\mathscr{S}$, then its CCFT $\hat{\phi}$ also belongs to $\mathscr{S}$ an vice-versa. It allows us to define the CCFT of generalized signals of polynomi: growth as follows.

C.49. Definition: The generalized CCFT.

(a) *CCFT of a tempered distribution.* The generalized CCFT of the tempere distribution $\mathbf{x}$ is the tempered distribution $\hat{\mathbf{x}}$ defined by

$$\hat{\mathbf{x}}(\phi) = \mathbf{x}(\hat{\phi}) \qquad \text{for all} \quad \phi \in \mathscr{S},$$

with $\hat{\phi} \in \mathscr{S}$ the CCFT of ϕ.

(b) *CCFT of a tempered generalized signal.* Suppose that the tempered distribution $\mathbf{x}$ is represented by the generalized signal x. Then the generalized CCFT $\hat{x}$ of x is the tempered generalized signal that represents the CCFT $\hat{\mathbf{x}}$ of $\mathbf{x}$. ■

As usual with operations on generalized signals, the CCFT of a generalized signal is defined by its "effect" on suitable test functions. For regular signals the definition reduces to the usual one.

C.50. Consistency proof. We prove that if $x \in \mathscr{L}_2$ is a regular signal, then the definition C.49 of the CCFT of x results in the usual CCFT. The regular signal x defines the tempered distribution $\mathbf{x}$ given by

$$\mathbf{x}(\phi) = \int_{-\infty}^{\infty} x(t)\phi(t)\, dt, \qquad \phi \in \mathscr{S}.$$

According to C.49, the CCFT $\hat{\mathbf{x}}$ of $\mathbf{x}$ is the distribution given by

$$\hat{\mathbf{x}}(\phi) = \mathbf{x}(\hat{\phi}) = \int_{-\infty}^{\infty} x(t)\hat{\phi}(t)\, dt = \int_{-\infty}^{\infty} x(t)\left(\int_{-\infty}^{\infty} \phi(f)e^{-j2\pi ft}\, df\right) dt, \qquad \phi \in \mathscr{S}.$$

Note that when writing $\hat{\phi}$ as the CCFT of ϕ for convenience we interchanged the usual roles of f and t. We now apply *Fubini's theorem,* which states, roughly, that the order of a repeated integral may be reversed provided the result is a convergent integral. This yields

$$\begin{aligned}\hat{\mathbf{x}}(\phi) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t)\phi(f)e^{-j2\pi ft}\,df\right)dt\\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t)\phi(f)e^{-j2\pi ft}\,dt\right)df\\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt\right)\phi(f)\,df\\ &= \int_{-\infty}^{\infty} \hat{x}(f)\phi(f)\,df, \qquad \phi \in \mathcal{S}.\end{aligned}$$

This confirms that the generalized CCFT $\hat{\mathbf{x}}$ of $\mathbf{x}$ is a regular tempered distribution represented by the regular CCFT $\hat{x}$. ∎

Guido Fubini (1897–1943) was an Italian mathematician who taught in Turin until he emigrated to the United States in 1938.

By way of example, we consider the generalized CCFT of some signals that have no CCFT in the ordinary sense.

C.51. Example: Generalized CCFTs.
(a) *CCFT of the δ-function.* Suppose that

$$x(t) = \delta(t), \qquad t \in \mathbb{R}.$$

Then the generalized CCFT $\hat{\mathbf{x}}$ of the tempered distribution $\mathbf{x} = \boldsymbol{\delta}$ is defined by

$$\begin{aligned}\hat{\mathbf{x}}(\phi) = \mathbf{x}(\hat{\phi}) &= \int_{-\infty}^{\infty} \delta(t)\hat{\phi}(t)\,dt = \hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(f)\,df\\ &= \int_{-\infty}^{\infty} 1\cdot\phi(f)\,df, \qquad \phi \in \mathcal{S}.\end{aligned}$$

This shows that $\hat{\mathbf{x}}$ is a regular tempered distribution represented by

$$\hat{x}(f) = 1, \qquad f \in \mathbb{R}.$$

Thus, the CCFT of the δ-function is the constant 1.

(b) *CCFT of a constant.* Next, suppose that x is the constant signal

$$x(t) = 1, \qquad t \in \mathbb{R},$$

which has no CCFT in the ordinary sense. The generalized CCFT $\hat{\mathbf{x}}$ of the tempered distribution $\mathbf{x}$ represented by x is given by

$$\hat{\mathbf{x}}(\phi) = \mathbf{x}(\hat{\phi}) = \int_{-\infty}^{\infty} 1\cdot\hat{\phi}(f)\,df = \int_{-\infty}^{\infty} \hat{\phi}(f)\,df, \qquad \phi \in \mathcal{S}.$$

The latter expression simply is the inverse CCFT of $\hat{\phi}$ evaluated at 0, so that

$$\hat{\mathbf{x}}(\phi) = \phi(0) = \int_{-\infty}^{\infty} \delta(f)\phi(f)\,df, \qquad \phi \in \mathcal{S}.$$

This shows that the CCFT of the constant signal x is the δ-function

$$\hat{x}(f) = \delta(f), \qquad f \in \mathbb{R}.$$ ∎

C.52. Exercise: CCFT of derivatives of δ-functions. Prove that the CCFT of the kth derivative delta function $\delta^{(k)}$ is the regular tempered signal $(j2\pi f)^k$, $f \in \mathbb{R}$. ∎

The *inverse* CCFT of a tempered signal is easily established, namely by reversing Definition C.49.

C.53. Summary: Inverse generalized CCFT.
(a) *Inverse CCFT of a tempered distribution.* The inverse generalized CCFT of the tempered distribution $\hat{\mathbf{x}}$ is the tempered distribution $\mathbf{x}$ defined by

$$\mathbf{x}(\phi) = \hat{\mathbf{x}}(\check{\phi}) \qquad \text{for all} \quad \phi \in \mathcal{S},$$

with $\check{\phi}$ the inverse CCFT of ϕ.

(b) *Inverse CCFT of a tempered generalized signal.* Suppose that the tempered distribution $\hat{\mathbf{x}}$ is represented by the tempered generalized signal $\hat{x}$. Then the generalized inverse CCFT x of $\hat{x}$ is the tempered generalized signal that represents the inverse CCFT $\mathbf{x}$ of $\hat{\mathbf{x}}$. ∎

By way of example, we establish that application of the inverse CCFT to the CCFTs found in Example C.51 restores the original time signals.

C.54. Examples: Inverse generalized CCFT.
(a) *Inverse CCFT of a constant.* Suppose that $\hat{x}$ is the constant signal

$$\hat{x}(f) = 1, \qquad f \in \mathbb{R}.$$

By C.53, the inverse CCFT x of the tempered distribution $\hat{\mathrm{x}}$ defined by $\hat{x}$ is given by

$$\mathrm{x}(\phi) = \hat{\mathrm{x}}(\check{\phi}) = \int_{-\infty}^{\infty} \hat{x}(f)\check{\phi}(f)\, df = \int_{-\infty}^{\infty} \check{\phi}(f)\, df, \qquad \phi \in \mathcal{S}.$$

The latter expression is the CCFT ϕ of $\check{\phi}$ evaluated at 0, so that

$$\mathrm{x}(\phi) = \phi(0) = \int_{-\infty}^{\infty} \delta(t)\phi(t)\, dt, \qquad \phi \in \mathcal{S}.$$

This proves that the inverse CCFT of the constant 1 is the δ-function, in agreement with C.51(a), where we found that the CCFT of the δ-function is the constant 1.

(b) *Inverse CCFT of the δ-function.* By C.53, the inverse CCFT x of the tempered distribution $\hat{\mathrm{x}} = \boldsymbol{\delta}$ is given by

$$\mathrm{x}(\phi) = \int_{-\infty}^{\infty} \delta(f)\check{\phi}(f)\, df = \check{\phi}(0) = \int_{-\infty}^{\infty} \phi(t)\, dt, \qquad \phi \in \mathcal{S}.$$

This confirms that the inverse CCFT of the δ-function is the constant 1, as expected from C.51(b), where we found that the CCFT of the constant 1 is the δ-function. ■

C.55. Exercise: Inverse CCFT of derivatives of δ-functions. Show by application of the inverse CCFT that for every nonnegative integer k the generalized CCFT of the tempered signal $(-j2\pi t)^k$, $t \in \mathbb{R}$, is $\delta^{(k)}(f), f \in \mathbb{R}$. ■

The generalized CCFT possesses all the important properties the ordinary CCFT has, to wit, the *linearity, convolution, shift,* and *differentiation* properties and their converses. Also the symmetry properties carry over.

The identities of *Plancherel* and *Parseval* hold of course for regular signals that belong to $\mathcal{L}_2$, but do not make sense for regular signals not in $\mathcal{L}_2$ or for singular signals, because singular signals cannot always be multiplied and never can be squared.

The convolution property, which implies that convolution transforms into a product, and vice-versa, only applies if the convolutions and products actually exist.

C.56. Example: CCFT of a harmonic signal. We apply the shift property to determine the CCFT of the harmonic signal

$$z(t) = e^{j2\pi f_o t}, \qquad t \in \mathbb{R},$$

with f_o a real number. The shift property implies that if the CCFT of x is $\hat{x}$, the CCFT of $x(t)e^{j2\pi\phi t}, f \in \mathbb{R}$, is $\hat{x}(f - \phi), f \in \mathbb{R}$. Writing

$$z(t) = e^{j2\pi f_o t} \cdot 1, \qquad t \in \mathbb{R},$$

and recalling from C.51(b) that the CCFT of the constant 1 is $\delta(f), f \in \mathbb{R}$, it follows that the CCFT of the harmonic z is

$$\hat{z}(f) = \delta(f - f_o), \qquad f \in \mathbb{R}.$$

The interpretation of this result is that a complex harmonic with frequency f_o has all its frequency content concentrated at f_o. ■

Supplement D

Jordan Normal Form

The discussion of the modal transformation of linear time-invariant state difference or differential systems of Section 5.6 is based on the assumption that the $N \times N$ matrix A has N linearly independent eigenvectors. If A does not have N linearly independent eigenvalues (i.e., A is defective), then it cannot be diagonalized, but it can be transformed into a near-diagonal form, called the *Jordan normal form*. We summarize the main results.

D.1. Summary: Jordan normal form. Suppose that the $N \times N$ matrix A has L mutually different eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_L$, such that the eigenvalue λ_i has multiplicity N_i in the characteristic polynomial $\det(\lambda I - A)$. Then there exists a nonsingular $N \times N$ matrix V, which may be partitioned into L blocks of columns as

$$V = [V_1, V_2, \cdots, V_L],$$

such that

$$V^{-1}AV = J.$$

J consists of L diagonal blocks

$$J = \operatorname{diag}(J_1, J_2, \cdots, J_L).$$

For each $i \in \{1, 2, \cdots, L\}$ the block V_i has N_i columns and the block J_i has dimensions $N_i \times N_i$. The block J_i may be subpartitioned into L_i diagonal sub-blocks, with $L_i \leq N_i$, as

$$J_i = \operatorname{diag}(J_{i1}, J_{i2}, \cdots, J_{iL_i}), \tag{1}$$

where each sub-block J_{ij} is of the form

$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}.$$

J is called the *Jordan normal form* of the matrix A. ■

> Camille Jordan (1838–1922) was a French mathematician who taught at the Ecole Polytechnique in Paris.

In the case of a defective matrix, the Jordan normal form J takes the place of the diagonal matrix Λ with the eigenvalues of A as diagonal elements. The Jordan normal form also has the eigenvalues of A on the main diagonal, but on the first diagonal above the main diagonal it has sequences of ones interrupted by zeros. An example of a matrix that is in Jordan form, with two sub-blocks, is

$$J = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right].$$

From $J = V^{-1}AV$ it follows that

$$AV = VJ.$$

This suggests the following way of determining the transformation matrix V and the Jordan normal form of A. Denote the columns of V as $v_1, v_2, \cdots, v_N$. Then, from the form of J it follows with $AV = VJ$ that

$$Av_j = \lambda v_j + \gamma_j v_{j-1}, \qquad j = 2, 3, \cdots, N, \tag{2}$$

where λ is an eigenvalue of A, and γ_j is either 0 or 1, depending on whether or not the ith column of J has a 1 above the main diagonal.

Let us subpartition the block V_i of V corresponding to the subpartitioning (1) of J_i as

$$V_i = [V_{i1}, V_{i2}, \cdots, V_{iL_i}].$$

Then γ_i is zero whenever the corresponding column v_i is the *first* column of a sub-block.

If $\gamma_i = 0$ the vector v_i satisfies $Av_i = \lambda v_i$ and, hence, is an eigenvector of A. Thus, the *first* columns of the sub-blocks of V_i may be found by determining the eigenvectors of A corresponding to the eigenvalue λ. The remaining columns of V successively follow by (2) with $\gamma_j = 1$. These remaining columns are known as the *generalized eigenvectors* of the matrix A. An eigenvector v_i followed by a number of generalized eigenvectors satisfying (2) with $\gamma_j = 1$ is called a *chain*. We illustrate the procedure with an example.

D.2. Example: Jordan normal form. Consider the matrix A as obtained in Example 5.6.12(b) but suppose that $R_1 = R_2 = 1$, $C_1 = C_2 = 2$, and $L_1 = L_2 = \frac{1}{2}$. The matrix then takes the form

$$A = \begin{bmatrix} -2 & \frac{1}{2} & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & \frac{1}{2} \\ 0 & 0 & -2 & 0 \end{bmatrix},$$

and its characteristic polynomial is

$$\det(\lambda I - A) = (\lambda^2 + 2\lambda + 1)(\lambda^2 + 2\lambda + 1) = (\lambda + 1)^4.$$

Hence, the matrix A has the single eigenvalue -1 with multiplicity 4. It is easily verified that corresponding to this eigenvalue A has two linearly independent eigenvectors, which may be chosen as col(1, 2, 0, 0) and col(0, 0, 1, 2). Thus, choose the first column of V as

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

Then the second column v_2 of V by (2) must satisfy

$$Av_2 = -v_2 + v_1.$$

It may be found that this equation has the solution

$$v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

To this solution one may add any multiple of v_1, but we leave it as it is. It may also be verified that

$$Av_3 = -v_3 + v_2$$

has no solution so that the first chain consisting of an eigenvector followed by generalized eigenvectors consists of v_1 and v_2.

We now start a new chain by setting v_3 equal to the second eigenvector we found, namely,

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Solution of

$$Av_4 = -v_4 + v_3$$

results in

$$v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

This completes the computation of V, which together with its inverse takes the form

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}, \qquad V^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & \frac{1}{2} \end{bmatrix}.$$

The Jordan normal form of A is

$$J = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Given the Jordan normal form of a matrix A it is comparatively simple to determine integral powers of A and its exponential, needed to compute transition matrices of linear time-invariant discrete- or continuous-time systems.

D.3. Summary: Powers and exponentials of matrices. Consider the matrix A as in D.1. Then,

(a) $A^n = VJ^nV^{-1}$ for all $n \in \mathbb{Z}_+$.
(b) $e^{At} = Ve^{Jt}V^{-1}$ for all $t \in \mathbb{R}$.
(c) $J^n = \operatorname{diag}(J_1^n, J_2^n, \cdots, J_L^n)$ and
$J_i^n = \operatorname{diag}(J_{i1}^n, J_{i2}^n, \cdots, J_{iL_i}^n)$ for all $n \in \mathbb{Z}_+$.
(d) $e^{Jt} = \operatorname{diag}(e^{J_1t}, e^{J_2t}, \cdots, e^{J_Lt})$ and
$e^{J_it}, = \operatorname{diag}(e^{J_{i1}t}, e^{J_{i2}t}, \cdots, e^{J_{iL_i}t})$ for all $t \in \mathbb{R}$.
(e) For all $n \in \mathbb{Z}_+$,

$$J_{ij}^n = \begin{bmatrix} \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \binom{n}{2}\lambda_i^{n-2} & \cdots & \binom{n}{N_{ij}-1}\lambda_i^{N_{ij}-1} \\ 0 & \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \cdots & \binom{n}{N_{ij}-2}\lambda_i^{N_{ij}-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_i^n \end{bmatrix},$$

where N_{ij} is the dimension of J_{ij}.
(f) For all $t \in \mathbb{R}$,

$$e^{J_{ij}t} = e^{\lambda_i t}\begin{bmatrix} 1 & t & \dfrac{t^2}{2!} & \cdots & \dfrac{t^{N_{ij}-1}}{(N_{ij}-1)!} \\ 0 & 1 & t & \cdots & \dfrac{t^{N_{ij}-2}}{(N_{ij}-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$ ∎

The computation of powers of A and its exponential reduces to the computation of powers and the exponential of its Jordan sub-blocks J_{ij}, for which simple formulas exist. In D.3, $\binom{n}{j}$ denotes the binomial coefficient

$$\binom{n}{j} = \begin{cases} \dfrac{n!}{j!(n-j)!} & \text{for } j = 0, 1, \cdots, n, \\ 0 & \text{otherwise.} \end{cases}$$

D.4. Example: Powers and exponential of a matrix. In Example D.2 we found the Jordan normal form J of a 4×4 matrix A as

$$J = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

By D.3 and by using the fact that $\binom{n}{n-1} = n$ we obtain the nth power of J as

$$J^n = \begin{bmatrix} (-1)^n & n(-1)^n & 0 & 0 \\ 0 & (-1)^n & 0 & 0 \\ 0 & 0 & (-1)^n & n(-1)^n \\ 0 & 0 & 0 & (-1)^n \end{bmatrix}, \qquad n \in \mathbb{Z}_+.$$

The exponential of Jt is

$$e^{Jt} = e^{-t}\begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad t \in \mathbb{R}.$$

We use the latter result to compute the exponential of the matrix At as

$$e^{At} = Ve^{Jt}V^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} e^{-t} \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & \frac{1}{2} \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} 1-t & t/2 & 0 & 0 \\ -2t & 1+t & 0 & 0 \\ 0 & 0 & 1-t & t/2 \\ 0 & 0 & -2t & 1+t \end{bmatrix}, \qquad t \in \mathbb{R}.$$ ∎

Supplement E

Proofs

This supplement contains proofs of various results in Chapters 4, 7, and 8.

Proofs for Chapter 4

The proof for Chapter 4 relate to BIBO and CICO stability.

E.1. Proof of 4.6.3: BIBO stability of initially-at-rest difference and differential systems.

Discrete-time case. By 4.5.1, the initially-at-rest difference system described by the constant coefficient differential equation $Q(\sigma)y = P(\sigma)u$ is a convolution system with impulse response h given by

$$h(n) = \sum_{i=1}^{N} \alpha_i y_i(n-1)\mathbb{1}(n-1) + \sum_{i=0}^{M-N} \beta_i \Delta(n+i), \qquad n \in \mathbb{Z}.$$

By 3.6.2, a convolution system is BIBO stable if and only if its impulse response has finite action $\|h\|_1$. Thus, for the BIBO stability of the initially-at-rest system we need to check when h has finite action. By 4.4.1, each basis solution y_i may be chosen to be of the form $n^i\lambda^n$, $n \in \mathbb{Z}_+$, where λ is a characteristic root of the system. If $|\lambda| < 1$, then the basis solution converges exponentially to zero. If *all* characteristic

roots have magnitude less than 1, then the impulse response h consists of expon tially converging terms only and, hence, has finite action, so that the initially-at-r system is BIBO stable.

Suppose that λ^* is a characteristic root that *cancels* against a root of P (w full multiplicity m^*). This means that λ^* is a characteristic root but *not* a pole of t system. We show that the basis solutions corresponding to this root do not appear the impulse response h and, hence, do not affect the BIBO stability of the initiall at-rest system. Define the polynomials Q^* and P^* as the polynomials Q and P, r spectively, with all common factors corresponding to the root λ^* *canceled*. Then t difference system $Q^*(\sigma)y = P^*(\sigma)u$ has a uniquely defined impulse response h^* which does not contain any terms corresponding to the root λ^*. Now

$$Q^*(\sigma)h^* = P^*(\sigma)\Delta$$

implies

$$(\sigma - \lambda^*)^{m^*}Q^*(\sigma)h^* = (\sigma - \lambda^*)^{m^*}P^*(\sigma)\Delta,$$

which in turns implies $Q(\sigma)h^* = P^*(\sigma)\Delta$. Since h^* satisfies the initial conditions for the initially-at-rest system, it follows that h^* is the impulse response of the initially-at-rest system, so that $h^* = h$. This proves that h does not contain any terms corresponding to the root λ^*.

Thus, a sufficient condition for the impulse response h to have finite action is that all the poles of the system have magnitude strictly less than 1. It may be shown that *all* poles have a nonzero corresponding term in the impulse response, so that the sufficient condition is also necessary.

Continuous-time case. By 4.5.1, the impulse response of the initially-at-rest differential system described by the constant coefficient differential equation $Q(D)y = P(D)u$ is of the form

$$h(t) = \sum_{i=1}^{N} \alpha_i y_i(t)\mathbb{1}(t) + \sum_{i=1}^{M-N} \beta_i \delta^{(i)}(t), \qquad t \in \mathbb{R}.$$

If the degree of P is greater than that of Q (i.e., $M - N > 0$), then the impulse response contains derivatives of delta functions and, hence, has infinite action. This is why for BIBO stability the degree of P should be less than or equal to that of Q. The proof why the poles of the system need have strictly negative real part follows the same line as that for the discrete-time case. Because the basis solutions are of the form $t^ie^{\lambda t}$, $t \in \mathbb{R}_+$, convergence is determined by the sign of the real part $\mathrm{Re}(\lambda)$ of the poles. ∎

E.2. Proof of 4.6.13: CICO stability of convolution systems. The proof of 4.6.13 is given for the discrete-time case. That for continuous-time systems is similar, with integrals replacing the sums.

Denoting the impulse response of the convolution system as h, the IO relationship of the system takes the form $y = h * u$. If (u_1, y_1) and (u_2, y_2) are any two IO pairs it follows by linearity that $y_1 - y_2 = h * (u_1 - u_2)$. Thus, writing $u_1 - u_2 =: \tilde{u}$ and $y_1 - y_2 =: \tilde{y}$ we have $\tilde{y} = h * \tilde{u}$. Hence, from the definition, the system is CICO stable if and only if (i) it is BIBO stable and (ii) $\|\tilde{u}\|_\infty < \infty$ and $\tilde{u}(t) \to 0$ imply that $\tilde{y}(t) \to 0$.

Necessity. BIBO stability obviously is a necessary condition for CICO stability.

Sufficiency. Suppose that the system is BIBO stable. Then we need prove that $\|\tilde{u}\|_\infty < \infty$ and $\tilde{u}(n) \to 0$ as $n \to \infty$ imply that $\tilde{y}(n) \to 0$. If $\tilde{u}(n) \to 0$, for every $\epsilon_1 > 0$ there exists an n_1 such that $|\tilde{u}(n)| \leq \epsilon_1$ for all $n \geq n_1$. It follows that

$$\begin{aligned}
|\tilde{y}(n)| &= \left|\sum_{k\in\mathbb{Z}} h(n-k)\tilde{u}(k)\right| \\
&= \left|\sum_{k<n_1} h(n-k)\tilde{u}(k) + \sum_{k\geq n_1} h(n-k)\tilde{u}(k)\right| \\
&\leq \sum_{k<n_1} |h(n-k)|\cdot|\tilde{u}(k)| + \sum_{k\geq n_1} |h(n-k)|\cdot|\tilde{u}(k)| \\
&\leq \sum_{k<n_1} |h(n-k)|\cdot\|\tilde{u}\|_\infty + \sum_{k\geq n_1} |h(n-k)|\cdot\epsilon_1, \qquad n \in \mathbb{Z}.
\end{aligned}$$

Substituting $n - k = m$ it follows that

$$\begin{aligned}
|\tilde{y}(n)| &\leq \sum_{m>n-n_1} |h(m)|\cdot\|\tilde{u}\|_\infty + \sum_{m\leq n-n_1} |h(m)|\cdot\epsilon_1 \\
&\leq \sum_{m>n-n_1} |h(m)|\cdot\|\tilde{u}\|_\infty + \|h\|_1\cdot\epsilon_1, \qquad n \in \mathbb{Z}. \qquad (1)
\end{aligned}$$

Because by assumption the system is BIBO stable, $\|h\|_1$ is finite by 3.6.2(a). It follows that for every $\epsilon_2 > 0$ there exists an n_2 such that

$$\sum_{m>n_2} |h(m)| \leq \epsilon_2 \qquad \text{for} \quad n \geq n_2.$$

Thus, from (1) we obtain

$$|\tilde{y}(n)| \leq \epsilon_2\cdot\|\tilde{u}\|_\infty + \|h\|_1\cdot\epsilon_1 \qquad \text{for} \quad n \geq n_1 + n_2,\ n \in \mathbb{Z}.$$

As a result, by choosing $n_1 + n_2$ large enough, $|\tilde{y}(n)|$ may be made as small as desired for $n \geq n_1 + n_2$, and hence $\tilde{y}(n)$ approaches 0 as n goes to infinity. This proves that if the convolution system is BIBO stable, then it is also CICO stable. ∎

Proofs for Chapter 7

We continue with the proofs of some of the properties of the various Fourier transforms of Chapter 7.

E.3. Proof of 7.3.6: Convolution and converse convolution properties of the DDFT and CDFT.

(a′) *Convolution property of the CDFT.* We prove the convolution property for the CDFT; that for the DDFT is similar. We may express the CDFT of the cyclical convolution $z = x \odot y$ of two continuous-time signals x and y defined on $[0, P)$ as

$$\hat{z}(f) = \int_0^P \left(\int_0^P x((t-\tau) \bmod P)y(\tau)\,d\tau\right)e^{-j2\pi ft}\,dt, \qquad f \in \mathbb{Z}(F),$$

with $F = 1/P$. Interchanging the order of integration we obtain

$$\hat{z}(f) = \int_0^P \left(\int_0^P x((t-\tau) \bmod P)e^{-j2\pi ft}\,dt\right)y(\tau)\,d\tau, \qquad f \in \mathbb{Z}(F). \qquad (2)$$

The integral inside the large parentheses may be written as

$$\begin{aligned}
&\int_0^P x((t-\tau) \bmod P)e^{-j2\pi ft}\,dt \\
&\qquad = \int_0^\tau x(t-\tau+P)e^{-j2\pi ft}\,dt + \int_\tau^P x(t-\tau)e^{-j2\pi ft}\,dt, \qquad 0 \leq \tau < P.
\end{aligned}$$

Substitution of $t - \tau + P = \theta$ into the first integral on the right-hand side and $t - \tau = \theta$ into the second together with the fact that fP is an integer results in

$$\begin{aligned}
&\int_0^P x((t-\tau) \bmod P)e^{-j2\pi ft}\,dt \\
&= \int_{-\tau+P}^P x(\theta)e^{-j2\pi f(\theta+\tau-P)}\,d\theta + \int_0^{P-\tau} x(\theta)e^{-j2\pi f(\theta+\tau)}\,d\theta \\
&= \left(\int_0^P x(\theta)e^{-j2\pi f\theta}\,d\theta\right)e^{-j2\pi f\tau} = \hat{x}(f)e^{-j2\pi f\tau}, \qquad 0 \leq \tau < P.
\end{aligned}$$

With this it follows from (2) that

$$\hat{z}(f) = \int_0^P \hat{x}(f)e^{-j2\pi f\tau}y(\tau)\,d\tau = \hat{x}(f)\hat{y}(f), \qquad f \in \mathbb{Z}(F),$$

which proves that circular convolution transforms into multiplication.

(b′) *Converse convolution property of the CDFT.* To show that multiplication transforms into convolution we work in reverse direction. Consider

$$
\begin{aligned}
(\hat{x} * \hat{y})(f) &= F \sum_{\phi \in \mathbb{Z}(F)} \hat{x}(f - \phi)\hat{y}(\phi) \\
&= F \sum_{\phi \in \mathbb{Z}(F)} \left(\int_0^P x(t) e^{-j2\pi(f-\phi)t} \, dt \right) \hat{y}(\phi), \qquad f \in \mathbb{Z}(F).
\end{aligned}
$$

Interchanging the order of summation and integration we obtain

$$
\begin{aligned}
(\hat{x} * \hat{y})(f) &= \int_0^P x(t) \left(F \sum_{\phi \in \mathbb{Z}(F)} \hat{y}(\phi) e^{j2\pi\phi t} \right) e^{-j2\pi f t} \, dt \\
&= \int_0^P x(t) y(t) e^{-j2\pi f t} \, dt, \qquad f \in \mathbb{Z}(F),
\end{aligned}
$$

which shows that $\hat{x} * \hat{y}$ is the CDFT of $x \cdot y$. ∎

E.4. Proof of 7.3.7: Shift and converse shift properties of the DDFT and CDFT.

(a′) *Shift property of the CDFT.* The shift property is only proved for the CDFT. The proof for the DDFT is analogous. The shift property may easily be proved by simple substitution. Consider the CDFT of z given by

$$
z(t) = x((t + \theta) \bmod P), \qquad t \in [0, P).
$$

We have

$$
\hat{z}(f) = \int_0^P x((t + \theta) \bmod P) e^{-j2\pi f t} \, dt, \qquad f \in \mathbb{Z}(F),
$$

with $F = 1/P$. Suppose for the time being that $0 \le \theta < P$. Then substitution of $t + \theta = \tau$ yields

$$
\begin{aligned}
\hat{z}(f) &= \int_\theta^{\theta+P} x(\tau \bmod P) e^{-j2\pi f(\tau-\theta)} \, d\tau, \\
&= e^{j2\pi f\theta} \left(\int_\theta^P x(\tau) e^{-j2\pi f\tau} \, d\tau + \int_P^{\theta+P} x(\tau - P) e^{-j2\pi f\tau} \, d\tau \right), \qquad f \in \mathbb{Z}(F).
\end{aligned}
$$

Substitution of $\tau = t$ in the first term on the right-hand side and $\tau - P = t$ in the second yields, together with the fact that fP is an integer and, hence, $e^{j2\pi fP} = 1$,

$$
\begin{aligned}
\hat{z}(f) &= e^{j2\pi f\theta} \left(\int_\theta^P x(t) e^{-j2\pi f t} \, dt + \int_0^\theta x(t) e^{-j2\pi f t} \, dt \right) \\
&= e^{j2\pi f\theta} \hat{x}(f), \qquad f \in \mathbb{Z}(F).
\end{aligned}
$$

This proves the shift property. If θ does not lie in the interval $[0, P)$ one may wri $\theta = kP + \theta'$, with k an integer and $\theta' \in [0, P)$, and repeat the proof with min modification.

(b′) *Converse shift property of the CDFT.* The CDFT of the signal z given t

$$
z(t) = e^{j2\pi\phi t} x(t), \qquad t \in [0, P),
$$

with $\phi \in \mathbb{Z}(F)$, is

$$
\begin{aligned}
\hat{z}(f) &= \int_0^P e^{-j2\pi f t} e^{j2\pi\phi t} x(t) \, dt = \int_0^P e^{-j\pi(f-\phi)t} x(t) \, dt \\
&= \hat{x}(f - \phi), \qquad f \in \mathbb{Z}(F).
\end{aligned}
$$

This proves the converse shift property. ∎

E.5. Proof of 7.3.9: Differentiation property of the CDFT. Suppose that the time signal x defined on the finite time axis $[0, P)$ is cyclically continuous and has derivative $z = Dx$. It follows by partial integration that the CDFT $\hat{z}$ of the derivative is given by

$$
\begin{aligned}
\hat{z}(f) &= \int_0^P Dx(t) e^{-j2\pi f t} \, dt = x(t) e^{-j2\pi f t} \Big|_0^P + j2\pi f \int_0^P x(t) e^{-j2\pi f t} \, dt \\
&= j2\pi f \cdot \hat{x}(f), \qquad f \in \mathbb{Z}(F).
\end{aligned}
$$

This proves the differentiation property. ∎

E.6. Derivation of the DCFT. The derivation of the DCFT follows by reversing the roles of time and frequency in the CDFT. Let $z \in \mathcal{L}_2[0, P)$ be a finite-energy signal defined on the time axis $[0, P)$. Then by the CDFT we have

$$
z(t) = F \sum_{f \in \mathbb{Z}(F)} \hat{z}(f) e^{j2\pi f t}, \qquad t \in [0, P),
$$

where $F = 1/P$ and $\hat{z} \in \ell_2(F)$ is given by

$$
\hat{z}(f) = \int_0^P z(t) e^{-j2\pi f t} \, dt, \qquad f \in \mathbb{Z}(F).
$$

Interchanging f and t on the one hand and z and $\hat{z}$ on the other, we obtain

$$\hat{z}(f) = F \sum_{t \in \mathbb{Z}(F)} z(t) e^{j2\pi f t}, \qquad f \in [0, P),$$

$$z(t) = \int_0^P \hat{z}(f) e^{-j2\pi f t}\, df, \qquad t \in \mathbb{Z}(F).$$

The latter equality shows that the signal z, which belongs to $\ell_2(F)$ and, hence, has finite energy, may be expanded in a continuum of harmonics with frequencies ranging over a finite interval. It remains to rearrange so that we obtain the result in the form of 7.4.1. Setting $P = 1$ and hence $F = 1$, substituting $z(t) = x(-t)$, $t \in \mathbb{Z}$, and replacing $\hat{z}$ with $\hat{x}$ results in

$$\hat{x}(f) = \sum_{t \in \mathbb{Z}} x(-t) e^{j2\pi f t}, \qquad f \in [0, 1),$$

$$x(-t) = \int_0^1 \hat{x}(f) e^{-j2\pi f t}\, df, \qquad t \in \mathbb{Z}.$$

The DCFT and its inverse now follow by replacing t with $-n$ in both expressions, so that

$$\hat{x}(f) = \sum_{n \in \mathbb{Z}} x(n) e^{-j2\pi f n}, \qquad f \in [0, 1),$$

$$x(n) = \int_0^1 \hat{x}(f) e^{j2\pi f n}\, df, \qquad n \in \mathbb{Z}. \qquad \blacksquare$$

Proofs for Chapter 8

In the remainder of this supplement we present the proofs of several of the properties of the z- and Laplace transforms as listed in Section 8.4. Most of the proofs are given either for the discrete- or for the continuous-time case. The proof for the alternate case in each instance is closely parallel, with integrals replacing sums or vice-versa.

E.7. Proof of 8.4.2: Convolution property of the z- and Laplace transforms. We prove the convolution properties of the two- and one-sided Laplace transforms.

(a′) *Convolution property of the two-sided Laplace transform.* Let x and y be continuous-time signals whose convolution $w = x * y$ exists. The two-sided Laplace transform W of z is given by

$$W(s) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t-\tau) y(\tau)\, d\tau \right) e^{-st}\, dt.$$

We invoke *Fubini's theorem,* which states that if a repeated integral exists, the order of integration may be reversed. It follows that

$$W(s) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t-\tau) y(\tau) e^{-st}\, dt \right) d\tau$$
$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t-\tau) e^{-st}\, dt \right) y(\tau)\, d\tau.$$

Substitution of $t - \tau = \theta$ in the inside integral results in

$$W(s) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\theta) e^{-s\theta}\, d\theta \right) y(\tau) e^{-s\tau}\, d\tau = X(s) \int_{-\infty}^{\infty} y(\tau) e^{-s\tau}\, d\tau$$
$$= X(s) \cdot Y(s),$$

which proves the convolution property for the two-sided Laplace transform. For W to exist we need both X and Y to exist, so that the existence region of W is the intersection of the existence regions of X and Y. If the intersection is empty, the convolution does not exist.

(b′) *Convolution property of the one-sided Laplace transform.* Suppose that x and y are continuous-time signals, and let $w = x\mathbb{1} * y\mathbb{1}$. Since the supports of $x\mathbb{1}$ and $y\mathbb{1}$ are both contained in $[0, \infty)$, by 3.5.5 the support of w is also contained in $[0, \infty)$, so that w is zero for negative times. By the convolution property of the two-sided Laplace transform,

$$W = \mathscr{L}(x\mathbb{1}) \cdot \mathscr{L}(y\mathbb{1}) = X_+ \cdot Y_+,$$

where W is the two-sided Laplace transform of w. Because w is zero for negative times, its one-sided Laplace transforms W_+ equals its two-sided transform W. It follows that

$$W_+ = X_+ \cdot Y_+,$$

which proves the convolution property for the one-sided Laplace transform. $\blacksquare$

E.8. Proof of 8.4.3: Shift properties of the z- and Laplace transforms. We prove the shift properties for the discrete-time case.

(a) *Shift property of the two-sided z-transform.* The two-sided z-transform of the shifted discrete-time signal

$$w(n) = x(n + k), \qquad n \in \mathbb{Z},$$

is

$$W(z) = \sum_{n=-\infty}^{\infty} x(n + k)z^{-n}.$$

Substitution of $n + k = m$ results in

$$W(z) = \sum_{m=-\infty}^{\infty} x(m)z^{-(m-k)} = \left(\sum_{m=-\infty}^{\infty} x(m)z^{-m}\right)z^k$$
$$= z^k X(z),$$

where X is the two-sided z-transform of x. W exists if X exists, so that the existence region of W coincides with that of X.

(b) *Shift property of the one-sided z-transform.* The one-sided z transform of the backward shifted signal

$$w(n) = x(n + 1), \qquad n \in \mathbb{Z},$$

is

$$W_+(z) = \sum_{n=0}^{\infty} x(n + 1)z^{-n}.$$

By substitution of $n + 1 = m$ it follows that

$$W_+(z) = \sum_{m=1}^{\infty} x(m)z^{-(m-1)} = \left(\sum_{m=0}^{\infty} x(m)z^{-m} - x(0)\right)z$$
$$= zX_+(z) - zx(0),$$

where X_+ is the one-sided z-transform of x.

Similarly, the one-sided z-transform of the forward shifted signal

$$v(n) = x(n - 1), \qquad n \in \mathbb{Z},$$

is

$$V_+(z) = \sum_{n=0}^{\infty} x(n - 1)z^{-n}.$$

By the substitution $n - 1 = m$ we obtain

$$V_+(z) = \sum_{m=-1}^{\infty} x(m)z^{-(m+1)} = x(-1) + \left(\sum_{m=0}^{\infty} x(m)z^{-m}\right)z^{-1}$$
$$= z^{-1}X_+(z) + x(-1).$$

The existence regions of W_+ and V_+ are the same as that of X_+. This completes t proof of the shift properties of the one-sided z transform.

(c) *Converse shift property of the z-transform.* The two-sided z transform the signal

$$w(n) = a^n x(n), \qquad n \in \mathbb{Z},$$

is

$$W(z) = \sum_{n=-\infty}^{\infty} a_n x(n)z^{-n} = \sum_{n=-\infty}^{\infty} x(n)\left(\frac{z}{a}\right)^{-n}$$
$$= X\left(\frac{z}{a}\right).$$

$W(z)$ exists for those z such that z/a is in the existence region of X.

The proof of the converse shift property of the one-sided z-transform is essentially the same. ∎

E.9. Proof of 8.4.5: Differentiation properties of the Laplace transform.

(a) *Differentiation property of the two-sided Laplace transform.* The two-sided Laplace transform of the derivative $w = Dx$ of the continuous-time signal x is

$$W(s) = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-st}\, dt.$$

It follows by partial integration that

$$W(s) = x(t)e^{-st}\Big|_{-\infty}^{\infty} + s\int_{-\infty}^{\infty} x(t)e^{-st}\, dt$$
$$= sX(s),$$

where X is the two-sided Laplace transform of X. The product $x(t)e^{-st}$ vanishes at $t = \pm\infty$ because of the assumed existence of $X(s)$. The existence region of W coincides with that of X.

(b) *Differentiation property of the one-sided Laplace transform.* By partial integration it follows that the one-sided Laplace transform of the derivative $w = Dx$ of x is

$$W_+(s) = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st}\, dt = x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st}\, dt$$
$$= sX_+(s) - x(0^-), \qquad s \in \mathscr{E}_+,$$

where X_+ is the one-sided Laplace transform of x. The existence region of W_+ is the same as that of X_+. ∎

E.10. Proof of 8.4.7: Converse differentiation property of the z- and Laplace transforms. We prove the converse differentiation property of the two-sided z-transform. Suppose that X is the two-sided z-transform of x with existence region $\mathscr{E}$. Term-by-term differentiation of

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}, \qquad z \in \mathscr{E},$$

results in

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n)(-n)z^{-n-1}, \qquad z \in \mathscr{E},$$

so that

$$z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} [-nx(n)]z^{-n}, \qquad z \in \mathscr{E}.$$

The proof of the converse differentiation property of the one-sided z-transform is similar. ∎

E.11. Proof of 8.4.8: Integration properties of the Laplace transform.

(a) *Integration property of the two-sided Laplace transform.* Because

$$(\mathbb{1} * x)(t) = \int_{-\infty}^{\infty} \mathbb{1}(t-\tau)x(\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, d\tau, \qquad t \in \mathbb{R},$$

it follows that the integrated signal y defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau, \qquad t \in \mathbb{R},$$

may be expressed as

$$y = \mathbb{1} * x.$$

Application of the convolution property of the two-sided Laplace transform results in

$$Y(s) = \frac{1}{s} \cdot X(s) = \frac{X(s)}{s}.$$

Because the existence region of the Laplace transform $1/s$ of the unit step is $\text{Re}(s) > 0$, the existence region of the Laplace transform Y of y is $\{s \in \mathscr{E} \mid \text{Re}(s) > 0\}$, with $\mathscr{E}$ the existence region of x.

(b) *Integration property of the one-sided Laplace transform.* Consider the signal y defined by

$$y(t) = \int_{-\infty}^{t} (x \cdot \mathbb{1})(\tau)\, d\tau = \begin{cases} 0 & \text{for } t < 0, \\ \int_0^t x(\tau)\, d\tau & \text{for } t \geq 0, \end{cases} \qquad t \in \mathbb{R}.$$

Note that because y is zero for negative times its one-sided Laplace transform Y_+ equals its two-sided Laplace transform Y. Also, the two-sided Laplace transform of $x \cdot \mathbb{1}$ is the one-sided transform X_+ of x. By application of the integration property of the two-sided Laplace transform it thus follows that

$$Y_+(s) = \frac{\mathscr{L}(x\mathbb{1})(s)}{s} = \frac{X_+(s)}{s}, \qquad s \in \{s \in \mathscr{E}_+ \mid \text{Re}(s) > 0\},$$

where $\mathscr{E}_+$ is the existence region of X_+. ∎

E.12. Proof of 8.4.9: Initial and final value properties of the one-sided z- and Laplace transforms.

(a) *Initial value property of the one-sided z-transform.* To prove the initial value property, we write the one-sided z-transform X_+ of x as

$$X_+(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots, \qquad |z| > \rho,$$

with ρ the positive number that determines the existence region of X_+. Letting $|z|$ approach ∞ yields the desired result.

(a′) *Initial value property of the one-sided Laplace transform.* In the continuous-time case we make the initial value property plausible as follows. We have

$$sX_+(s) = s\int_0^{\infty} x(t)e^{-st}\, dt, \qquad \text{Re}(s) > \sigma, \tag{3}$$

with σ the real number that determines the existence region of the one-sided Laplace transform X_+ of x. If s is very large and real, the function $se^{-st}\mathbb{1}(t)$, $t \in \mathbb{R}$, may be considered an approximation to the δ-function, so that in the limit $s \to \infty$ the right-hand side of (3) approaches $x(0^+)$.

(b) *Final value property of the one-sided z-transform.* The complete proof of the final value property is beyond the scope of this text. Consider the discrete-time case. Given the signal x, the idea of the proof is to define another signal $\tilde{x}$ as

$$\tilde{x}(n) = x(n) - x(\infty), \qquad n \in \mathbb{Z},$$

that is, $\tilde{x}$ is the signal x with the limit $x(\infty)$ subtracted. The result is that $\tilde{x}$ approaches zero as time goes to infinity. The one-sided z-transform $\tilde{X}$ of x is given by

$$\tilde{X}_+(z) = X_+(z) - \frac{z}{z-1}x(\infty), \qquad |z| > 1,$$

where we use the assumption that X_+ exists for $|z| > 1$. It follows that

$$zx(\infty) = (z-1)X(z) - (z-1)\tilde{X}(z).$$

The difficult part of the proof is to show that because $\tilde{x}(n) \to 0$ as $n \to \infty$, $\tilde{X}(z)$ is finite at $z = 1$. Once this has been established, the final value property follows easily by letting z approach 1. ∎

Bibliography

The following books roughly cover the same material as the present text. The list is by no means complete.

R. A. Gabel and R. A. Roberts, *Signals and Linear Systems*, 3rd. ed., John Wiley, New York, 1987.

T. H. Glisson, *Introduction to System Analysis*. McGraw-Hill, New York, 1985.

C. L. Liu and J. W. S. Liu, *Linear Systems Analysis*. McGraw-Hill, New York, 1975.

H. P. Neff, Jr., *Continuous and Discrete Linear Systems*. Harper and Row, New York, 1984.

A. V. Oppenheim and A. S. Willsky, with I. T. Young, *Signals and Systems*. Prentice Hall, Englewood Cliffs, N.J., 1983.

H. W. Schüssler, *Netzwerke, Signale und Systeme*, vols. 1 and 2. Springer-Verlag, Berlin, 1981, 1984.

W. M. Siebert, *Circuits, Signals, and Systems*. MIT Press, Cambridge, Mass., 1986.

More advanced texts on signals and systems follow:

T. Kailath, *Linear Systems*. Prentice Hall, Englewood Cliffs, N.J., 1980.

A. Papoulis, *Signal Analysis*. McGraw-Hill, New York, 1977.